

# PURITY FOR FAMILIES OF GALOIS REPRESENTATIONS

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**ABSTRACT.** We formulate a notion of purity for  $p$ -adic big Galois representations and pseudorepresentations of Weil groups of  $\ell$ -adic number fields for  $\ell \neq p$ . This is obtained by showing that all powers of the monodromy of any big Galois representation stay “as large as possible” under pure specializations. The role of purity for families in the study of the variation of local Euler factors, local automorphic types along irreducible components, the intersection points of irreducible components of  $p$ -adic families of automorphic Galois representations is illustrated using the examples of Hida families and eigenvarieties. Moreover, using purity for families, we improve a part of the local Langlands correspondence for  $\mathrm{GL}_n$  in families formulated by Emerton and Helm.

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## 1. INTRODUCTION

**1.1. Motivation.** Let  $r$  be a geometric Galois representation of the absolute Galois group of a number field with coefficients in  $\overline{\mathbb{Q}}_p$ . Then the restriction  $r_v$  of  $r$  to a decomposition group at any given finite place  $v$  not dividing  $p$  is potentially unipotent by Grothendieck’s monodromy theorem (see [ST68, p. 515–516]). Given a projective smooth variety  $X$  over a finite extension  $K$  of  $\mathbb{Q}_\ell$ , the weight-monodromy conjecture ([Ill94, Conjecture 3.9]) says that for any prime  $p \neq \ell$  and any integer  $i \geq 0$ , the  $\text{Gal}(\overline{K}/K)$ -representation  $H_{\text{ét}}^i(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p)$  is pure of weight  $i$ , *i.e.*, the  $i$ -th shift of the associated monodromy filtration coincides with the associated weight filtration (see definition 2.10). When  $r$  is irreducible, the representation  $r_v$  is expected to be pure. The Galois representations attached to cuspidal automorphic representations (which are algebraic in the sense of [Clo90, Definition 1.8]) by the Langlands correspondence (which is often conjectural) provide ample examples of geometric representations. The purity of the restrictions of  $p$ -adic automorphic Galois representations to decomposition groups at places outside  $p$  is known in many cases due to works of Carayol [Car86], Harris, Taylor [HT01], Blasius [Bla06], Taylor, Yoshida [TY07], Shin [Shi11], Caraiani [Car12], Scholze [Sch12], Clozel [Clo13] *et. al.* Following works of Hida [Hid86a, Hid86b, Hid95], Mazur [Maz89], Coleman, Mazur [CM98], Chenevier [Che04], Bellaïche, Chenevier [BC04] *et. al.*, automorphic Galois representations are believed to live in  $p$ -adic families. Thus it is desirable to have a notion of purity for families. The goal of this article is to provide a formulation of this notion and to discuss its applications to  $p$ -adic families of Galois representations.

**1.2. Purity for families.** The most naive way to formulate purity for big Galois representations would be to relate the monodromy filtration with the weight filtration. However the Frobenius eigenvalues on a big Galois representation are elements of a ring of large Krull dimension and are not algebraic numbers in general, precluding the possibility of considering the weight filtration. Thus a formulation of purity for big Galois representations is not straightforward. On the other hand, it is natural to expect that such a formulation should include a compatibility statement at pure specializations.

This formulation is achieved in theorem 4.1, which we call *purity for big Galois representations* because it says that the structures of Frobenius-semisimplifications of Weil-Deligne parametrizations of pure specializations of a ( $p$ -adic) big Galois representation (of the Weil group of an  $\ell$ -adic number field with  $\ell \neq p$ ) are “rigid”. In other words, it says that given a pure Weil-Deligne representation, its lifts to Weil-Deligne representations over integral domains have the “same structure”.

An important example of families of Galois representations comes from eigenvarieties. The traces of the Galois representations attached to the arithmetic points of an eigenvariety are interpolated by a pseudorepresentation defined over the global sections of the eigenvariety. Thus a notion of purity for pseudorepresentations is indispensable for the understanding of various local properties of the arithmetic points of eigenvarieties. This is provided by theorem 5.4, which we call *purity for pseudorepresentations*. It says that given an  $\mathcal{O}$ -valued pseudorepresentation  $T$  of the Weil group of an  $\ell$ -adic number field (where  $\mathcal{O}$  is a characteristic zero domain over  $\mathbb{Z}_p$  with  $p \neq \ell$ ), the Frobenius-semisimplification of two Weil-Deligne representations over two domains (containing  $\mathcal{O}$  as a subalgebra) have the “same structure” if their traces are equal to  $T$  and each of them has a pure specialization. This is deduced using purity for big Galois representations.

By [BC09, Lemma 7.8.11], around each nonempty admissible open affinoid subset  $U$ , the pseudorepresentation defined over the global sections of an eigenvariety lifts to a Galois representation on a finite type module over some integral extension of the normalization of  $\mathcal{O}(U)$ . But this module is not known to be free over its coefficient ring. This forbids us from applying theorem 5.4 to eigenvarieties. To circumvent this problem, we prove a general result in theorem 5.6 which we explain now. Let  $T : G_F \rightarrow \mathcal{O}$  (where  $\mathcal{O}$  is a characteristic zero domain over  $\mathbb{Z}_p$  and  $F$  is a number field) be a pseudorepresentation which is equal to  $T_1 + \cdots + T_n$  where  $T_1 : G_F \rightarrow \mathcal{O}, \dots, T_n : G_F \rightarrow \mathcal{O}$  are traces of some irreducible  $G_F$ -representations over  $\overline{\mathbb{Q}}(\mathcal{O})$  whose restrictions to the Weil group of a finite place  $w \nmid p$  of  $F$  are monodromic (see definition 2.2). Let  $\mathcal{O}, \mathcal{O}'$  be two domains containing  $\mathcal{O}$  as a subalgebra. Let  $\mathfrak{p}$  (resp.  $\mathfrak{p}'$ ) be a prime ideal of  $\mathcal{O}$  (resp.  $\mathcal{O}'$ ) such that the residue field  $\kappa = \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$  (resp.  $\kappa' = \mathcal{O}'_{\mathfrak{p}'}/\mathfrak{p}'\mathcal{O}'_{\mathfrak{p}'}$ ) is a finite extension of  $\mathbb{Q}_p$  and the Henselization  $\mathcal{O}_{\mathfrak{p}}^h$  (resp.  $\mathcal{O}'_{\mathfrak{p}'}^h$ ) of  $\mathcal{O}_{\mathfrak{p}}$  (resp.  $\mathcal{O}'_{\mathfrak{p}'}$ ) is Hausdorff. Suppose  $T_1 \bmod \mathfrak{p}, \dots, T_n \bmod \mathfrak{p}$  (resp.  $T_1 \bmod \mathfrak{p}', \dots, T_n \bmod \mathfrak{p}'$ ) are traces of irreducible  $G_F$ -representations  $\rho_1, \dots, \rho_n$  (resp.  $\rho'_1, \dots, \rho'_n$ ) over  $\overline{\kappa}$  (resp.  $\overline{\kappa}'$ ) and the representations  $\rho_1|_{G_w}, \dots, \rho_n|_{G_w}$  (resp.  $\rho'_1|_{G_w}, \dots, \rho'_n|_{G_w}$ ) are pure. Then using [Nys96, Théorème 1] and purity for pseudorepresentations, we show in theorem 5.6 that the structure of  $\mathrm{WD}(\oplus_{i=1}^n \rho_i|_{W_w})^{\mathrm{Fr-ss}}$  and  $\mathrm{WD}(\oplus_{i=1}^n \rho'_i|_{W_w})^{\mathrm{Fr-ss}}$  are “rigid”. Thus theorem 5.6 can be applied to eigenvarieties to prove the “rigidity” of the Frobenius-semisimplifications of the Weil-Deligne parametrizations of the local Galois representations attached to the arithmetic points that lie within the “irreducibility and purity locus” (see definition 5.5) of (certain pseudorepresentations attached to) pseudorepresentations defined over global sections of eigenvarieties. Henceforth, by *purity for families*, we refer to theorem 4.1, 5.4, 5.6.

**1.3. Statement of purity for big Galois representations.** In theorem 1.1 below, we state a special case of theorem 4.1. We refer to §5 for the statements of theorem 5.4, 5.6.

Let  $p, \ell$  be two distinct primes and  $K$  denote a finite extension of  $\mathbb{Q}_{\ell}$ . Denote the absolute Galois group of  $K$  by  $G_K$ . Let  $I_K$  denote the inertia group and  $W_K$  denote the Weil group. Let  $q$  denote the cardinality of the residue field  $k$  of the ring of integers  $\mathcal{O}_K$  of  $K$ . Fix an element  $\phi \in G_K = \mathrm{Gal}(\overline{K}/K)$  which lifts the geometric Frobenius  $\mathrm{Fr}_k \in G_k = \mathrm{Gal}(\overline{k}/k)$ . The Frobenius-semisimplification of the Weil-Deligne parametrization of a monodromic (see definition 2.2) representation  $V$  of  $W_K$  is denoted by  $\mathrm{WD}(V)^{\mathrm{Fr-ss}}$ . We refer to §1.4.1 and §1.6 for few more notations. From now on by a *big Galois representation*, we mean a monodromic representation  $\rho : W_K \rightarrow \mathrm{Aut}_{\mathcal{R}}(\mathcal{T})$  of  $W_K$  on a free  $\mathcal{R}$ -module  $\mathcal{T}$  of finite rank where  $\mathcal{R}$  is a domain containing  $\mathbb{Z}_p$  as a subalgebra. Note that if  $\mathcal{R}$  is a local ring with finite residue field and  $\rho|_{I_K}$  is continuous, then  $\rho$  is monodromic by Grothendieck’s monodromy theorem (see [ST68, p. 515–516]). Also if  $\mathcal{R}$  is an affinoid algebra over  $\mathbb{Q}_p$  and  $\rho|_{I_K}$  is continuous, then  $\rho$  is monodromic by Grothendieck’s monodromy theorem (see [BC09, Lemma 7.8.14]). Denote the  $W_K$ -representation  $\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathbb{Q}}(\mathcal{R})$  by  $\mathcal{V}$  and let

$$\mathrm{WD}(\mathcal{V})^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\chi_i \otimes \rho_i)_{/\overline{\mathbb{Q}}(\mathcal{R})}$$

be the isomorphism of Weil-Deligne representations (as in equation (4.1)) where  $m, t_1 \leq t_2 \leq \cdots \leq t_m$  are positive integers,  $\chi_1, \dots, \chi_m$  are  $(\mathcal{R}^{\mathrm{intal}})^{\times}$ -valued unramified characters of  $W_K$  and  $\rho_1, \dots, \rho_m$  are irreducible Frobenius-semisimple representations of  $W_K$  over  $\mathcal{R}^{\mathrm{intal}}[1/p]$  with finite image. Given a field  $E$  and a ring homomorphism  $f : \mathcal{R} \rightarrow E$ , the

$W_K$ -representation  $\mathcal{T} \otimes_{\mathcal{R},f} E$  is denoted by  $V_f$ . We fix an isomorphism  $\iota_p : \overline{\mathbb{Q}_p} \simeq \mathbb{C}$  and let  $\text{rec}$  denote the reciprocity map as in §3.

**Theorem 1.1** (Purity for big Galois representations). *Let  $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}_p}$  be a  $\mathbb{Z}_p$ -algebra homomorphism such that  $V_\lambda$  is pure. Then the following hold.*

- (1) *The rank of no power of the monodromy of  $\mathcal{T}_p$  decreases after specializing at  $\lambda$ .*
- (2) *The Weil-Deligne representations  $\text{WD}(V_\lambda)^{\text{Fr-ss}}$  and  $\oplus_{i=1}^m \text{Sp}_{t_i}(\lambda^{\text{intal}} \circ (\chi_i \otimes \rho_i))_{/\overline{\mathbb{Q}_p}}$  are isomorphic.*
- (3) *The polynomial  $\text{Eul}(\mathcal{V}, X)^{-1}$  has coefficients in  $\mathcal{R}^{\text{int}}$  and*

$$(1.1) \quad \lambda(\text{Eul}(\mathcal{V}, X)^{-1}) = \text{Eul}(V_\lambda, X)^{-1}.$$
- (4) *If  $\xi : \mathcal{R} \rightarrow \overline{\mathbb{Q}_p}$  is a  $\mathbb{Z}_p$ -algebra homomorphism such that  $V_\xi$  is pure, then the automorphic types of  $\text{rec}(\iota_p(\text{WD}(V_\xi)^{\text{Fr-ss}}))$  and  $\text{rec}(\iota_p(\text{WD}(V_\lambda)^{\text{Fr-ss}}))$  are the same.*

Moreover, for any field extension  $\mathcal{K}$  of  $\mathbb{Q}_p$  and any  $\mathbb{Z}_p$ -algebra homomorphism  $\mu : \mathcal{R} \rightarrow \mathcal{K}$  with  $\lambda(\ker \mu) = 0$ , the Weil-Deligne representation  $\text{WD}(V_\mu \otimes_{\mathcal{K}} \overline{\mathcal{K}})^{\text{Fr-ss}}$  is isomorphic to  $\oplus_{i=1}^m \text{Sp}_{t_i}(\mu^{\text{intal}} \circ (\chi_i \otimes \rho_i))_{/\overline{\mathcal{K}}}$ .

Note that when  $\mathcal{D} := \{\ker \lambda \mid \lambda \in \text{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}_p}), V_\lambda \text{ is pure}\}$  is dense in  $\text{Spec}(\mathcal{R})$ , using Hilbert's nullstellensatz, some of the above results (for example, equation (1.1)) can be proved for  $\lambda = \iota_p \circ \text{mod } \mathfrak{p}$  for  $\mathfrak{p}$  varying in a dense subset of  $\mathcal{D}$  (here  $\iota_p$  denotes a  $\mathbb{Z}_p$ -algebra injection from  $\mathcal{R}/\mathfrak{p}$  to  $\overline{\mathbb{Q}_p}$ ).

**1.4. Applications.** Theorem 4.1, 5.4, 5.6 turn out to be useful in the study of some arithmetic aspects of  $p$ -adic families of Galois representations. For example, the local Langlands correspondence for  $\text{GL}_n$  in families, the local automorphic types of arithmetic points of  $p$ -adic families, the geometry of the underlying spaces of families etc. These are studied in theorem 6.1, 7.2, 7.4, 7.6. In this section, we state a special case of theorem 6.1 and explain the content of theorem 7.2, 7.4, 7.6.

**1.4.1. Local Langlands correspondence for  $\text{GL}_n$  in families.** The local Langlands correspondence, proved by Harris, Taylor [HT01], asserts that there is a canonical bijection between the isomorphism classes of  $n$ -dimensional Frobenius-semisimple complex Weil-Deligne representation of  $W_K$  and the isomorphism classes of irreducible admissible representations of  $\text{GL}_n(K)$ . This is extended to  $p$ -adic families of representations of  $G_K$  by Emerton and Helm in [EH14]. We state a special case of it.

First, we fix some notations. Given a monodromic representation  $\rho : G_K \rightarrow \text{GL}_n(L)$  over a field  $L$  of characteristic zero, the representation attached to  $\text{WD}(\rho)^{\text{Fr-ss}}$  by the extension ([EH14, §4.2]) of the modified local Langlands correspondence of Breuil and Schneider (see [BS07, p. 161–164]) is denoted by  $\pi(\rho)$  and the smooth contragredient of  $\pi(\rho)$  is denoted by  $\tilde{\pi}(\rho)$  (the representations  $\pi(\rho), \tilde{\pi}(\rho)$  are equal to  $\pi(\text{WD}(\rho)^{\text{Fr-ss}}), \tilde{\pi}(\text{WD}(\rho)^{\text{Fr-ss}})$  respectively in the notation of [EH14]). Let  $A$  be a complete reduced  $p$ -torsion free Noetherian local ring with finite residue field of characteristic  $p$ . The residue field of a prime ideal  $\mathfrak{p}$  of  $A$  is denoted by  $\kappa(\mathfrak{p})$ .

Given a continuous Galois representation  $r : G_K \rightarrow \text{GL}_n(A)$ , there exists at most one admissible smooth  $\text{GL}_n(K)$ -representation  $V$  over  $A$ , up to isomorphism, satisfying some conditions (conditions (1), (2), (3) of [EH14, Theorem 1.2.1]). Suppose such a  $V$  exists. Let  $\mathcal{D}$  denote the set of primes  $\mathfrak{p}$  of  $A[1/p]$  such that the number of irreducible components of

$\mathrm{Spec}A[1/p]$  passing through  $\mathfrak{p}$  is one. By [EH14, Theorem 6.2.5], for any  $\mathfrak{p} \in \mathcal{D}$ , there is a  $\mathrm{GL}_n(K)$ -equivariant surjection

$$(1.2) \quad \tilde{\pi}(\kappa(\mathfrak{p}) \otimes_A r) \rightarrow \kappa(\mathfrak{p}) \otimes_A V.$$

Let  $\mathcal{D}'$  denote the set of primes  $\mathfrak{p}$  in  $\mathcal{D}$  for which the above map is an isomorphism. Then  $\mathcal{D}'$  contains an open dense subset  $U'$  of  $\mathrm{Spec}A[1/p]$  by [EH14, Theorem 1.2.1]. By theorem 1.2 below,  $\mathcal{D}'$  contains all the elements of  $\mathcal{D}$  that are contained in kernel of pure specializations. For a more general result, we refer to theorem 6.1 which is proved using [EH14, Theorem 6.2.1, 6.2.5, 6.2.6] and theorem 4.1.

**Theorem 1.2.** *Let  $V, \mathcal{D}, \mathcal{D}'$  be as above. Suppose  $V$  exists. Let  $\mathfrak{p}$  be a prime in  $\mathcal{D}$ . Suppose that there exists a  $\mathbb{Z}_p$ -algebra homomorphism  $i_{\mathfrak{p}} : A/\mathfrak{p} \rightarrow \overline{\mathbb{Q}_p}$  such that  $r \otimes_A A/\mathfrak{p} \otimes_{A/\mathfrak{p}, i_{\mathfrak{p}}} \overline{\mathbb{Q}_p}$  is pure. Then  $\mathfrak{p}$  lies in  $\mathcal{D}'$ .*

Hida's theory of ordinary automorphic representations provide continuous representations of absolute Galois group of number fields with coefficients in rings of the form  $A$ . So their restriction to decomposition groups at places not dividing  $p$  gives representations of the form  $r$ , to which [EH14, Theorem 6.2.1, 6.2.5, 6.2.6] and theorem 6.1 apply. On the other hand, overconvergent forms also form families, although of rather different nature, for instance, there are examples of such families whose coefficient rings are not local (and there are also families of overconvergent forms defined over local rings, see [AIS13]). The local Langlands correspondence is not yet extended to families defined over non-local rings or to the case when  $A$  is an affinoid algebra. However, the coefficient rings  $\mathcal{R}, \mathcal{O}, \mathcal{O}', \mathcal{O}''$  as in theorem 4.1, 5.4, 5.6 are quite general, for instance,  $\mathcal{R}, \mathcal{O}$  are not assumed to be local or Noetherian. So once a notion of local Langlands correspondence for more general families is established, it is likely that one could use theorem 4.1 5.4, 5.6 to show that the extension (as in [EH14, §4.2]) of the Breuil-Schneider modified local Langlands correspondence is interpolated at all the primes contained in the kernel of pure specializations.

**1.4.2. Hida families and eigenvarieties.** Given a  $p$ -adic family of Galois representations of the absolute Galois group of a number field, the variation of the Frobenius-semisimplifications of the Weil-Deligne parametrizations of the local Galois representations attached to the members at places outside  $p$  can be studied using theorem 4.1, 5.4, 5.6. Thus purity for families illustrates the variation of local Euler factors of the arithmetic points of  $p$ -adic families of automorphic Galois representations and also the variation of local automorphic types of arithmetic points when local-global compatibility is known. In §7, we explain this variation using the examples of Hida family of cusp forms, Hida family of ordinary automorphic representations of definite unitary groups, eigenvariety for definite unitary groups. We refer to theorem 7.2, 7.4, 7.6 for the precise statements. Roughly speaking, these three results state that the ‘‘Galois types’’ of the local Galois representations attached to the arithmetic points of any given irreducible component of these families are constant (under some hypotheses). In the proofs of theorem 7.2, 7.4, 7.6, we do not use the fact that the arithmetic points of these families form a dense subset. Moreover in theorem 7.2, we do not assume that the residual representation attached to (a branch of) the Hida family of ordinary cusp forms is residually absolutely irreducible. However in theorem 7.4, we only consider those branches of the Hida family (of ordinary automorphic representations of a definite unitary group) whose associated minimal primes are contained in non-Eisenstein maximal ideals. In

theorem 7.6, we assume that each irreducible component of the eigenvariety attached to the definite unitary group  $U(m)$  contains at least one arithmetic point such that its associated automorphic representation  $\pi$  is regular at infinity, the semisimple conjugacy class of  $\pi_p$  has  $m$  distinct eigenvalues and the weak base change of  $\pi$  is cuspidal. We assume further that the Galois representation attached to an automorphic representation  $\pi$  of  $U(m)$  of regular weight at infinity is irreducible if the weak base change of  $\pi$  is cuspidal. For related results, we refer to [Nek06, §12.7.14], [BC09, §7.5.3, 7.8.4], [Pau11, Theorem A].

**1.5. Organization.** The main results obtained in this article are theorem 4.1, 5.4, 5.6, 6.1, 7.2, 7.4, 7.6.

In §2, we introduce the notion of Weil-Deligne representations over domains following [Del73b, 8.4–8.6], [Tay04, p.77–78]. Then we recall the notion of pure representations and Euler factors. We begin section 3 by fixing a unitary local Langlands correspondence. Then we introduce the modified local Langlands correspondence of Breuil-Schneider ([BS07, p. 161–164]) and the extension of this modification due to Emerton and Helm ([EH14, §4.2]). In §3.2, we recall the formulation of the local Langlands correspondence for  $GL_n$  in  $p$ -adic families by Emerton and Helm. The notion of automorphic type is defined in §3.3.

In the next section, we prove theorem 4.1. In its proof, we crucially use (through equation (4.1) for instance) the hypothesis that the ring  $\mathcal{R}$  is a domain. We cannot expect to prove theorem 4.1 when the ring  $\mathcal{R}$  is replaced by a more general ring, an example being a ring with finitely many minimal primes. In fact a crucial step in its proof is to express the trace of  $\mathcal{V}$  as a sum of traces of irreducible Frobenius-semisimple representations over  $\mathcal{R}^{\text{intal}}$  and then to pin down the factors of powers of the character  $|\text{Art}_K^{-1}|_K$  in them. The amount of these factors is governed by the size of the Jordan blocks of the monodromy of  $\mathcal{V}$ . When the coefficient ring  $\mathcal{R}$  of  $\mathcal{T}$  is not a domain, then the shapes of the Jordan blocks of the images of its monodromy in the stalks of  $\text{Spec}(\mathcal{R})$  at the generic points need not be independent of the generic points. Thereby, in no reasonable manner, it is possible to pin down the factors of powers of  $|\text{Art}_K^{-1}|_K$  present in the representations stated above. Even in the very simple case where  $\mathcal{R} = \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]]$ ,  $\mathcal{V}$  is semistable and  $N \in M_3(\mathcal{R})$  is the strictly upper triangular matrix with  $N_{12} = (X, 0, 0)$ ,  $N_{13} = 0$ ,  $N_{23} = (0, X - 1, 0)$ , we cannot track the ‘right’ factors of powers of  $q$  in the characteristic roots of  $\phi$  on  $\mathcal{V}$ . Thus it seems hard to have a reasonable analogue of equation (4.1) that could lead to an analogue of theorem 4.1 when  $\mathcal{R}$  is a more general ring than a domain. So we are compelled to assume that  $\mathcal{R}$  is a domain.

In §5, we use purity for big Galois representations to prove purity for pseudorepresentations (theorem 5.4) by an induction argument. Theorem 5.6 follows as a corollary of theorem 5.4. Section 6 uses results from [EH14] and theorem 4.1 to prove theorem 6.1 about local Langlands correspondence for  $GL_n$  in families. In section 7, using the examples of Hida family of ordinary cusp forms, Hida family of ordinary automorphic representations of definite unitary groups and eigenvarieties, we illustrate the role of theorem 4.1, 5.4, 5.6 in the study of the local data (*eg.* local Euler factors, local automorphic types, Weil-Deligne representations) associated to the members of  $p$ -adic families of automorphic Galois representations (theorem 7.2, 7.4, 7.6).

**1.6. Notations.** For every field  $F$ , we fix an algebraic closure  $\overline{F}$  of it. For any finite place  $v$  of a number field  $E$ , the decomposition group  $\text{Gal}(\overline{E}_v/E_v)$  is denoted by  $G_v$ . Let  $W_v \subset G_v$

(resp.  $I_v \subset G_v$ ) denote the Weil group (resp. inertia group) and  $\text{Fr}_v \in G_v/I_v$  denote the geometric Frobenius element. We fix embeddings  $\mathbb{C} \xleftarrow{i_\infty} \overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$  once and for all. The largest reduced quotient of a ring  $A$  is denoted by  $A_{\text{red}}$  and the map  $A_{\text{red}} \rightarrow B_{\text{red}}$  induced by a ring homomorphism  $f : A \rightarrow B$  is denoted by  $f_{\text{red}}$ . The fraction field of a domain  $A$  is denoted by  $Q(A)$  and the field  $\overline{Q(A)}$  is denoted by  $\overline{Q}(A)$ . If  $R$  is a ring with a unique minimal prime ideal, then the integral closure of  $R_{\text{red}}$  in  $Q(R_{\text{red}})$  (resp.  $\overline{Q}(R_{\text{red}})$ ) is denoted by  $R^{\text{int}}$  (resp.  $R^{\text{intal}}$ ). If  $f : R \rightarrow S$  is a ring homomorphism where  $S$  is a ring with a unique minimal prime ideal, then the map  $f_{\text{red}}$  has an extension to a map  $R^{\text{intal}} \rightarrow S^{\text{intal}}$ . We fix one such map and denote it by  $f^{\text{intal}}$ . If an integer  $m$  is nonzero in  $S^{\text{red}}$ , then the unique extension  $R^{\text{intal}}[1/m] \rightarrow S^{\text{intal}}[1/m]$  of  $f^{\text{intal}}$  is also denoted by  $f^{\text{intal}}$ . By a representation of a group  $G$  on a module  $M$  over a ring  $A$ , we mean a group homomorphism  $G \rightarrow \text{Aut}_A(M)$  (even if  $G$  is a topological group) unless otherwise stated.

## 2. LOCAL GALOIS REPRESENTATIONS

Let  $\varpi$  denote a uniformizer of  $\mathcal{O}_K$  and  $\text{val}_K : K^\times \rightarrow \mathbb{Z}$  be the  $\varpi$ -adic valuation. Let  $|\cdot|_K := (\#k)^{-\text{val}_K(\cdot)}$  be the corresponding norm. The Weil group  $W_K$  is defined as the subgroup of  $G_K$  consisting of elements which map to an integral power of  $\text{Fr}_k$  in  $G_k$ . The Artin map  $\text{Art}_K : K^\times \xrightarrow{\sim} W_K^{\text{ab}}$  is normalized so that the uniformizing parameters go to the lifts of the geometric Frobenius element. Let  $P_K \subset I_K$  denote the wild inertia subgroup. Then given a compatible system  $\zeta = (\zeta_n)_{\ell \nmid n}$  of primitive roots of unity, we have an isomorphism  $t_\zeta : I_K/P_K \xrightarrow{\sim} \prod_{p \neq \ell} \mathbb{Z}_p$  such that  $\sigma(\varpi^{1/n}) = \zeta_n^{(t_\zeta(\sigma) \bmod n)} \varpi^{1/n}$  for all  $\sigma \in I_K/P_K$ . By [NSW08, Theorem 7.5.2], for all  $\sigma \in W_K$  and  $\tau \in I_K$ , we have  $t_\zeta(\sigma\tau\sigma^{-1}) = \varepsilon(\sigma)t_\zeta(\tau)$  where  $\varepsilon := \prod_{p \neq \ell} \varepsilon_p : G_K \rightarrow \prod_{p \neq \ell} \mathbb{Z}_p^\times$  is the product of the cyclotomic characters. For a prime  $p \neq \ell$ , let  $t_{\zeta,p} : I_K \rightarrow \mathbb{Z}_p$  denote the composition of the projection  $I_K \rightarrow I_K/P_K$ , the map  $t_\zeta$  and the projection from  $\prod_{p \neq \ell} \mathbb{Z}_p$  to  $\mathbb{Z}_p$ . Define  $v_K : W_K \rightarrow \mathbb{Z}$  by  $\sigma|_{K^{\text{ur}}} = \text{Fr}_k^{v_K(\sigma)}$  for all  $\sigma \in W_K$ .

**Definition 2.1** ([Del73b, 8.4.1], [Tay04, p.77–78]). *Let  $A$  be a commutative domain of characteristic zero.*

- (1) *A Weil-Deligne representation of  $W_K$  on a free  $A$ -module  $M$  of finite rank is a triple  $(r, M, N)$  consisting of a representation  $r : W_K \rightarrow \text{Aut}_A(M)$  and a nilpotent endomorphism  $N \in \text{End}_A(M)$  such that  $r(I_K)$  is finite and for all  $\sigma \in W_K$ ,*

$$r(\sigma)Nr(\sigma)^{-1} = q^{-v_K(\sigma)}N$$

*in  $\text{End}_{A[1/\ell]}(M \otimes_A A[1/\ell])$ . The operator  $N$  is called the monodromy of  $(r, M, N)$ .*

- (2) *A representation  $\rho$  of  $W_K$  on a free module  $M$  of finite rank over a domain  $A$  is said to be irreducible Frobenius-semisimple if  $M \otimes \overline{Q}(A)$  is irreducible, the action of  $\phi$  on  $M \otimes \overline{Q}(A)$  is semisimple and  $\#\rho(I_K) < \infty$ .*

The sum of Weil-Deligne representations are defined in the usual way (see [BH06, §31.2] for instance).

**Definition 2.2.** *Let  $A$  be a  $\mathbb{Z}_p$ -algebra of characteristic zero. Suppose  $M$  be an  $A$ -module together with a  $W_K$ -action  $\rho : W_K \rightarrow \text{Aut}_A(M)$  on it. We say  $M$  is monodromic with*

monodromy  $N$  over  $K'$  if there exists a finite extension  $K'/K$  and  $N$  is a nilpotent element of  $\text{End}_{A[1/p]}(M \otimes_A A[1/p])$  such that for all  $\tau \in I_{K'}$

$$\rho(\tau) = \exp(t_{\zeta,p}(\tau)N)$$

in  $\text{End}_{A[1/p]}(M \otimes_A A[1/p])$ . An  $A$ -module  $M'$  equipped with an action of  $G_K$  is said to be monodromic if  $M'|_{W_K}$  is monodromic.

Suppose  $(r, N) = (r, V, N)$  is a Weil-Deligne representation with coefficients in a field  $L$  of characteristic zero which contains all the characteristic roots of all the elements of  $r(W_K)$ . Let  $r(\phi) = r(\phi)^{ss}u = ur(\phi)^{ss}$  be the Jordan decomposition of  $r(\phi)$  as the product of a diagonalizable matrix  $r(\phi)^{ss}$  and a unipotent matrix  $u$ . Following [Del73b, 8.5], [Tay04, p. 78], define  $\tilde{r}(\sigma) = r(\sigma)u^{-v_K(\sigma)}$  for all  $\sigma \in W_K$ . Then  $(\tilde{r}, V, N)$  is a Weil-Deligne representation (by [Del73b, 8.5] for example) and is called the *Frobenius semisimplification* of  $(r, V, N)$  (cf. [Del73b, 8.6]). It is denoted by  $V^{\text{Fr-ss}}$ . We say  $(r, V, N)$  is *Frobenius-semisimple* if  $\tilde{r} = r$ .

**Definition 2.3.** For an integer  $t \geq 1$ , a characteristic zero commutative domain  $A$  with  $\ell \in A^\times$  and a representation  $(r, M)$  of  $W_K$  on a free module  $M$  of finite rank over  $A$  with  $\#r(I_K) < \infty$ , we denote by  $\text{Sp}_t(r)_{/A}$  the Weil-Deligne representation with underlying module  $M^t$  on which  $W_K$  acts via

$$r|\text{Art}_K^{-1}|_K^{t-1} \oplus r|\text{Art}_K^{-1}|_K^{t-2} \oplus \cdots \oplus r|\text{Art}_K^{-1}|_K \oplus r$$

and the monodromy  $N$  induces an isomorphism from  $r|\text{Art}_K^{-1}|_K^i$  to  $r|\text{Art}_K^{-1}|_K^{i+1}$  for all  $0 \leq i \leq t-2$  and is zero on  $r|\text{Art}_K^{-1}|_K^{t-1}$ .

Let  $\Omega$  denote an algebraically closed field of characteristic zero.

**Definition 2.4.** A Weil-Deligne representation over  $\Omega$  is said to be indecomposable if it is not isomorphic to a direct sum of two nonzero Weil-Deligne representations over  $\Omega$ .

**Theorem 2.5.** Let  $(\rho, V, N)$  be a Frobenius-semisimple Weil-Deligne representation over  $\Omega$ . Then it is isomorphic to

$$\bigoplus_{i \in I} \text{Sp}_{t_i}(r_i)_{/\Omega}$$

for some irreducible Frobenius-semisimple representations  $r_i : W_K \rightarrow \text{GL}_{n_i}(\Omega)$  and positive integers  $t_i$ . This decomposition is unique up to reordering and replacing factors by isomorphic factors.

*Proof.* This follows from the proof of [Del73a, Proposition 3.1.3 (i)].  $\square$

**Definition 2.6.** Let  $(\rho, V, N)$  be as above. Then the integer  $\max\{t_i | i \in I\}$  is called the size of  $\rho$ .

**Definition 2.7.** An indecomposable summand of a Frobenius-semisimple Weil-Deligne representation  $V$  over  $\Omega$  is a Weil-Deligne subrepresentation of  $V$  isomorphic to a summand  $\text{Sp}_{t_i}(r_i)_{/\Omega}$  via an isomorphism  $V \simeq \bigoplus_{i \in I} \text{Sp}_{t_i}(r_i)_{/\Omega}$  as in theorem 2.5.

While dealing with indecomposable summands of  $V$ , we always implicitly fix an isomorphism  $V \simeq \bigoplus_{i \in I} \text{Sp}_{t_i}(r_i)_{/\Omega}$  as in theorem 2.5.



**Proposition 2.8.** *Let  $(r, N)$  be a Weil-Deligne representation over an integral domain  $A$  of characteristic zero. Let  $\mathbb{Q}^{\text{cl}}$  denote the algebraic closure of  $\mathbb{Q}$  in  $\overline{\mathbb{Q}}(A)$ . Let  $B$  be a subring of  $A$  such that the characteristic polynomial of  $r(g)$  has coefficients in  $B$  for all  $g \in W_K$ . Then there exist*

- (i) *positive integers  $m, t_1 \leq \dots \leq t_m$ ,*
- (ii)  *$(B^{\text{intal}})^{\times}$ -valued unramified characters  $\chi_1, \dots, \chi_m$  of  $W_K$ ,*
- (iii) *irreducible Frobenius-semisimple representations  $\rho_1, \dots, \rho_m$  of  $W_K$  with coefficients in  $\mathbb{Q}^{\text{cl}}$  with finite image*

*such that  $((r, N) \otimes_A \overline{\mathbb{Q}}(A))^{\text{Fr-ss}}$  is isomorphic to  $\oplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \rho_i)$ .*

*Proof.* By theorem 2.5, there exist positive integers  $m, t_1 \leq t_2 \leq \dots \leq t_m$ , irreducible Frobenius-semisimple representations  $r_1, \dots, r_m$  of  $W_K$  over  $\overline{\mathbb{Q}}(A)$  such that  $((r, N) \otimes_A \overline{\mathbb{Q}}(A))^{\text{Fr-ss}}$  is isomorphic to  $\oplus_{i=1}^m \text{Sp}_{t_i}(r_i)$ . From the proof of [BH06, 28.6 Proposition], it follows that for each  $1 \leq i \leq m$ , there exists an unramified character  $\chi_i : W_K \rightarrow \overline{\mathbb{Q}}(A)^{\times}$  such that the  $W_K$ -representation  $\chi_i^{-1} \otimes r_i$  has finite image. So there exists an irreducible Frobenius-semisimple representation  $\rho_i : W_K \rightarrow \text{GL}_{d_i}(\mathbb{Q}^{\text{cl}})$  with finite image such that  $\chi_i^{-1} \otimes r_i$  and  $\rho_i$  are isomorphic over  $\overline{\mathbb{Q}}(A)$  (by [Tay91, Theorem 1] for instance). So the product of  $\chi_i(\phi)$  and a root of unity belongs to  $B^{\text{intal}}$ . Thus  $\chi_i(\phi)$  belongs to  $B^{\text{intal}}$  and similarly,  $\chi_i(\phi)^{-1}$  belongs to  $B^{\text{intal}}$ . Hence  $\chi_i$  has values in  $B^{\text{intal}}$ . This proves the result.  $\square$

**Lemma 2.9.** *Let  $r : W_K \rightarrow \text{GL}_n(A)$  be an irreducible Frobenius-semisimple representation of  $W_K$  with coefficients in a domain  $A$  of characteristic zero. If  $B$  is a domain and  $f : A \rightarrow B$  is a ring homomorphism, then  $f \circ r$  is also an irreducible Frobenius-semisimple representation.*

*Proof.* Let  $\mathbb{Q}^{\text{cl}}$  denote the algebraic closure of  $\mathbb{Q}$  in  $\overline{\mathbb{Q}}(A)$ . By proposition 2.8, there exist an unramified character  $\chi : W_K \rightarrow (A^{\text{intal}})^{\times}$  and an irreducible Frobenius-semisimple representation  $\rho : W_K \rightarrow \text{GL}_n(\mathbb{Q}^{\text{cl}})$  with finite image such that  $r$  is isomorphic to  $\chi \otimes \rho$  over  $\overline{\mathbb{Q}}(A)$ . As  $\rho(W_K)$  is finite, it is contained in  $\text{GL}_n(A^{\text{intal}}[1/m])$  for some positive integer  $m$ . So  $f^{\text{intal}}(\rho)$  is isomorphic to  $f^{\text{intal}}(\chi^{-1} \otimes r) = f^{\text{intal}}(\chi^{-1}) \otimes f^{\text{intal}}(r) = f^{\text{intal}}(\chi^{-1}) \otimes f(r)$ . Thus  $f(r)$  is isomorphic to  $f^{\text{intal}}(\chi) \otimes f^{\text{intal}}(\rho)$ . This proves the lemma.  $\square$

**Definition 2.10.** (cf. [Sch11, p.1014]) *A Frobenius-semisimple Weil-Deligne representation  $V$  of  $W_K$  over  $\overline{\mathbb{Q}}_p$  is said to be pure of weight  $w \in \mathbb{Z}$  if the eigenvalues of one (and hence any) lift of the geometric Frobenius element on  $\text{Gr}_i M_{\bullet}$  are  $q$ -Weil numbers of weight  $w + i$  where  $M_{\bullet}$  denotes the monodromy filtration on  $V$ .*

*A finite dimensional representation  $V$  of  $G_K$  or of  $W_K$  over  $\overline{\mathbb{Q}}_p$  is said to be pure of weight  $w \in \mathbb{Z}$  if  $V|_{W_K}$  is monodromic and the Frobenius semisimplification of the Weil-Deligne parametrization of  $V|_{W_K}$  with respect to one (and hence any) choice of  $\phi$  and  $\zeta$  is pure of weight  $w$ .*

We refer to [Mil94, Definition 2.5] for the notion of Weil numbers and to [Ill94, equation 1.5.5] for the notion of monodromy filtration.

**Remark 2.11.** Let  $r_1, \dots, r_m$  be irreducible Frobenius-semisimple representations of  $W_K$  over  $\overline{\mathbb{Q}}_p$ . Then by [Del80, I.6.7, p. 166], it follows that the Weil-Deligne representation  $\oplus_{i=1}^m \text{Sp}_{t_i}(r_i)_{/\overline{\mathbb{Q}}_p}$  is pure of weight  $w$  if and only if the  $\phi$ -eigenvalues on  $r_1|\text{Art}_K^{-1}|_K^{(t_1-1)/2}, \dots, r_m|\text{Art}_K^{-1}|_K^{(t_m-1)/2}$  are  $q$ -Weil numbers of weight  $w$  (for any choice of a square root of  $q$  in  $\overline{\mathbb{Q}}_p$ ).

Let  $\Omega$  be an algebraically closed field of characteristic zero. For a Weil-Deligne representation  $(r, V, N)$  of  $W_K$  over  $\Omega$ , its *Euler factor*  $\text{Eul}((r, N), X)$  is defined as the element  $\det(1 - X\phi|_{V^{I_K, N=0}})^{-1}$  of  $\Omega(X)$  where  $V^{I_K, N=0}$  denotes the subspace of  $V$  on which  $I_K$  acts trivially and  $N$  is zero. For a representation  $\rho : \text{Gal}(\overline{E}/E) \rightarrow \text{GL}(V)$  of the absolute Galois group of a number field  $E$  on a finite dimensional vector space  $V$  over  $\Omega$ , its local *Euler factor*  $\text{Eul}_v(\rho, X)$  at a finite place  $v$  of  $E$  not dividing  $p$  is defined to be the element  $\text{Eul}(\text{WD}(V|_{W_v}), X)$  in  $\Omega(X)$  if  $V|_{W_v}$  is monodromic. We refer to [Tay04, p. 85] for details.

### 3. THE LOCAL LANGLANDS CORRESPONDENCE AND ITS EXTENSIONS

The local Langlands correspondence for  $\text{GL}_n(K)$  is known due to works of Harris, Taylor [HT01]. Depending on the required normalization, there are various choices of this correspondence. We prefer to work with the unitary local Langlands correspondence, which depends on the choice of a square root of  $q$  in  $\overline{\mathbb{Q}}_p$ , which we fix from now on. We denote the reciprocity map by  $\text{rec}$ .

**3.1. The modified local Langlands correspondence of Breuil-Schneider.** We recall the modified local Langlands correspondence of Breuil-Schneider and its extension to Weil-Deligne representations with coefficients in any extension of  $\mathbb{Q}_p$ . We refer to [BS07, p. 161–164] and [EH14, §4.2] for details.

Let  $(\rho, N)$  be a Frobenius-semisimple Weil-Deligne representation of  $W_K$  over  $\overline{\mathbb{Q}}_p$ . Let  $\pi(\rho, N)$  denote the indecomposable admissible representation of  $\text{GL}_n(K)$  over  $K$  attached to  $(\rho, N)$  via the Breuil-Schneider modified local Langlands correspondence (see [BS07, p. 161–164]). To define the representation  $\pi(\rho, N)$ , one needs to choose a square root of  $q$ . However the representation  $\pi(\rho, N)$  is independent of this choice.

In [EH14], this modified correspondence is extended to Frobenius-semisimple Weil-Deligne representations over an arbitrary field extension of  $\mathbb{Q}_p$ . For a Frobenius-semisimple Weil-Deligne representation  $(\rho, N)$  of  $W_K$  over an extension  $L$  of  $\mathbb{Q}_p$ , let  $\pi(\rho, N)$  denote the indecomposable admissible representation of  $\text{GL}_n(K)$  over  $L$  attached to  $(\rho, N)$  (see §4.2 of *loc. cit.*). The smooth contragredient of  $\pi(\rho, N)$  is denoted by  $\tilde{\pi}(\rho, N)$ . If  $r$  is a monodromic representation of  $W_K$  on a finite dimensional vector space over  $L$ , then we denote by  $\tilde{\pi}(r)$  the representation  $\tilde{\pi}(\text{WD}(r)^{\text{Fr-ss}})$ .

**3.2. The local Langlands correspondence for  $\text{GL}_n$  in families.** Let  $A$  be a complete reduced  $p$ -torsion free Noetherian local ring with finite residue field of characteristic  $p$ . Let  $\mathfrak{m}$  denote the maximal ideal of  $A$ . The residue field of a prime ideal  $\mathfrak{p}$  of  $A$  is denoted by  $\kappa(\mathfrak{p})$ . For a prime ideal  $\mathfrak{p}$  of  $A$ , the mod  $\mathfrak{p}$  reduction of a representation  $\rho$  of a group on an  $A$ -module is denoted by  $\rho_{\mathfrak{p}}$ . We refer to [EH14] for unfamiliar notations and terminologies used below.

**Theorem 3.1.** *Let  $E$  be a number field and  $S$  denote a finite set of non-archimedean places of  $E$ , none of which lie over  $p$ . For each  $v \in S$ , let  $r_v : G_{E_v} \rightarrow \text{GL}_n(A)$  be a continuous representation. Write  $G = \prod_{v \in S} \text{GL}_n(E_v)$ . Then there exists at most one (up to isomorphism) admissible smooth representation  $V$  of  $G$  over  $A$  satisfying the conditions below.*

- (1)  $V$  is  $A$ -torsion free, i.e., all associated primes of  $V$  are minimal primes of  $A$ .

(2) For each minimal prime  $\mathfrak{a}$  of  $A$ , there is a  $G$ -equivariant isomorphism

$$\bigotimes_{v \in S} \tilde{\pi}(r_{v,\mathfrak{a}}) \xrightarrow{\sim} \kappa(\mathfrak{a}) \otimes_A V.$$

(3) The  $G$ -cosocle  $\text{cosoc}(V/\mathfrak{m}V)$  of  $V/\mathfrak{m}V$  is absolutely irreducible and generic, while the kernel of the natural surjection  $V/\mathfrak{m}V \rightarrow \text{cosoc}(V/\mathfrak{m}V)$  contains no generic subrepresentations.

*Proof.* It is a part of [EH14, Theorem 6.2.1].  $\square$

When  $V$  exists, we denote it by  $\tilde{\pi}(\{r_v\}_{v \in S})$ . When  $S$  contains only one place, we denote  $V$  by  $\tilde{\pi}(r_v)$ . By [EH14, Proposition 6.24], the  $A[G]$ -module  $\tilde{\pi}(\{r_v\}_{v \in S})$  exists if and only if each of the individual  $A[\text{GL}_n(E_v)]$ -modules  $\tilde{\pi}(r_v)$  exists. For a minimal prime  $\mathfrak{a}$  of  $A[1/p]$ , the monodromy of  $r_{v,\mathfrak{a}}$  is denoted by  $N_v(\mathfrak{a})$ .

**Theorem 3.2.** *Let  $S$  be as in theorem 3.1 and  $\mathfrak{p}$  be a prime ideal of  $A[1/p]$ . Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  be the minimal primes of  $A$  contained in  $\mathfrak{p}$ . For each  $i = 1, \dots, s$ , let  $V_i$  be the maximal  $A$ -torsion free quotient of  $\tilde{\pi}(\{r_v\}_{v \in S}) \otimes_A A/\mathfrak{a}_i$ . Let  $W_{\mathfrak{p}}$  denote the image of the diagonal map*

$$\kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\{r_v\}_{v \in S}) \rightarrow \prod_i \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{a}_i} V_i.$$

*Suppose that the  $A[G]$ -module  $\tilde{\pi}(\{r_v\}_{v \in S})$  exists. Then there is a  $\kappa(\mathfrak{p})$ -linear  $G$ -equivariant surjection*

$$\varsigma_{\mathfrak{p}} : \bigotimes_{v \in S} \tilde{\pi}(r_{v,\mathfrak{p}}) \rightarrow W_{\mathfrak{p}}.$$

*Moreover, if  $\mathfrak{a}$  is a minimal prime ideal of  $A$  contained in  $\mathfrak{p}$  such that the rank of  $N_v(\mathfrak{a})^j$  is equal to the rank of  $(N_v(\mathfrak{a}) \otimes_{A/\mathfrak{a}} \kappa(\mathfrak{p}))^j$  for all  $j \geq 1$  and for any  $v \in S$ , then the surjection  $\varsigma_{\mathfrak{p}}$  is an isomorphism. Furthermore, when  $s$  is equal to one, there is a  $\kappa(\mathfrak{p})$ -linear  $G$ -equivariant surjection*

$$\gamma_{\mathfrak{p}} : \bigotimes_{v \in S} \tilde{\pi}(r_{v,\mathfrak{p}}) \rightarrow \kappa(\mathfrak{p}) \otimes_A \tilde{\pi}(\{r_v\}_{v \in S})$$

*and it is an isomorphism if the rank of  $N_v(\mathfrak{a}_1)^j$  is equal to the rank of  $(N_v(\mathfrak{a}_1) \otimes_{A/\mathfrak{a}_1} \kappa(\mathfrak{p}))^j$  for all  $j \geq 1$  and for any  $v \in S$ .*

*Proof.* It is the content of [EH14, Theorem 6.2.5, 6.2.6].  $\square$

### 3.3. Automorphic types.

**Definition 3.3.** *Let  $(\rho, N)$  be a Frobenius-semisimple Weil-Deligne representation of  $W_K$  over a  $\overline{\mathbb{Q}}_p$ . Let  $I, m_1, \dots, m_I$  be positive integers and  $r_1, \dots, r_I$  be irreducible Frobenius-semisimple representations of  $W_K$  over  $\overline{L}$  such that  $(\rho, N) \otimes_L \overline{L}$  is isomorphic to  $\bigoplus_{i=1}^I \text{Sp}_{m_i}(r_i)$ . We define the automorphic representation type  $\text{AT}^{\text{rep}}(\text{rec}(\rho, N))$  of  $\text{rec}(\rho, N)$  to be*

$$\text{AT}^{\text{rep}}(\text{rec}(\rho, N)) = ((\text{rec}(r_1), m_1), \dots, (\text{rec}(r_I), m_I))$$

*and the automorphic type  $\text{AT}(\text{rec}(\rho, N))$  of  $\text{rec}(\rho, N)$  to be*

$$\text{AT}(\text{rec}(\rho, N)) = ((\dim r_1, m_1), \dots, (\dim r_I, m_I)).$$

Note that though automorphic representation type and automorphic type of  $\text{rec}(\rho, N)$  is defined using the ‘Galois data’  $r_i, m_i$  attached to  $(\rho, N)$ , these can also be defined in terms of automorphic representations attached to  $\text{rec}(\rho, N)$ . Thus these notions are automorphic in nature. In fact, if we use Bernstein-Zelevinsky classification [BZ77, Zel80] to express  $\text{rec}(\rho, N)$  as the quotient of an induced representation attached to some intervals  $[\pi_1, n_1], \dots, [\pi_J, n_J]$  where  $\pi_i$  is a supercuspidal representation of  $\text{GL}_{d_i}(K)$  (see [Rod82, §4.3] for details), then by the local Langlands correspondence (see [Rod82, §4.4] for instance), it follows that  $I = J$  and up to some reordering,  $\pi_i \simeq \text{rec}(r_i)$ ,  $d_i = \dim r_i$ ,  $n_i = m_i$  for all  $1 \leq i \leq I$ .

#### 4. PURITY FOR BIG GALOIS REPRESENTATIONS

Let  $K, \mathcal{R}, \mathcal{T}, \mathcal{V}, \iota_p$  be as in §1.3. Denote the fraction field of  $\mathcal{R}$  by  $\mathcal{L}$  and the algebraic closure of  $\mathbb{Q}$  in  $\overline{\mathcal{L}}$  by  $\mathbb{Q}^{\text{cl}}$ . Notice that  $\mathbb{Q}^{\text{cl}}$  is contained inside  $\mathcal{R}^{\text{intal}}[1/p]$ . Then by proposition 2.8, there exist positive integers  $m, t_1 \leq \dots \leq t_m$ , unramified characters  $\chi_1, \dots, \chi_m : W_K \rightarrow (\mathcal{R}^{\text{intal}})^{\times}$ , irreducible Frobenius-semisimple representations  $\rho_1 : W_K \rightarrow \text{GL}_{d_1}(\mathbb{Q}^{\text{cl}}), \dots, \rho_m : W_K \rightarrow \text{GL}_{d_m}(\mathbb{Q}^{\text{cl}})$  with finite image such that

$$(4.1) \quad \text{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \rho_i)_{/\overline{\mathcal{L}}}.$$

Let  $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}_p}$  be a  $\mathbb{Z}_p$ -algebra homomorphism and  $\pi_\lambda$  be the automorphic representation  $\text{rec}(\iota_p(\text{WD}(V_\lambda)^{\text{Fr-ss}}))$ .

**Theorem 4.1** (Purity for big Galois representations). *Suppose  $V_\lambda$  is pure of weight  $w$ .*

- (1) *The Weil-Deligne representations  $\text{WD}(V_\lambda)^{\text{Fr-ss}}$  and  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\lambda^{\text{intal}} \circ (\chi_i \otimes \rho_i))_{/\overline{\mathbb{Q}_p}}$  are isomorphic.*
- (2) *The rank of no power of the monodromy of  $\mathcal{T}_p$  decreases after specializing at  $\lambda$ .*
- (3) *The polynomial  $\text{Eul}(\mathcal{V}, X)^{-1}$  has coefficients in  $\mathcal{R}^{\text{int}}$  and its specialization under  $\lambda$  is  $\text{Eul}(V_\lambda, X)^{-1}$ .*
- (4) *The automorphic representation type  $\text{AT}^{\text{rep}}(\pi_\lambda)$  of  $\pi_\lambda$  is equal to*

$$((\text{rec}(\iota_p(\lambda^{\text{intal}}(\chi_1 \otimes \rho_1))), t_1), \dots, (\text{rec}(\iota_p(\lambda^{\text{intal}}(\chi_m \otimes \rho_m))), t_m)).$$

- (5) *The automorphic type  $\text{AT}(\pi_\lambda)$  of  $\pi_\lambda$  is equal to the unordered tuple  $\{(\dim \rho_1, t_1), \dots, (\dim \rho_m, t_m)\}$ .*

Moreover, for any field extension  $\mathcal{K}$  of  $\mathbb{Q}_p$  and any  $\mathbb{Z}_p$ -algebra homomorphism  $\mu : \mathcal{R} \rightarrow \mathcal{K}$  with  $\lambda(\ker \mu) = 0$ , the Weil-Deligne representation  $\text{WD}(V_\mu \otimes_{\mathcal{K}} \overline{\mathcal{K}})^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\mu^{\text{intal}} \circ (\chi_i \otimes \rho_i))_{/\overline{\mathcal{K}}}$ .

*Proof.* Denote the representation  $\chi_i \otimes \rho_i$  by  $r_i$  and the multiset  $\bigcup_{i=1}^m \{\lambda^{\text{intal}} \circ r_i, \lambda^{\text{intal}} \circ (|\text{Art}_K^{-1}|_K r_i), \dots, \lambda^{\text{intal}} \circ (|\text{Art}_K^{-1}|_K^{t_i-1} r_i)\}$  by  $S$ . Let  $N \in \text{End}_{\mathcal{R}_p}(\mathcal{T}_p)$  be the monodromy of  $\mathcal{T}_p$ . Note that conditions (A), (B), (C) below hold with  $D = t_m$  (by equation (4.1)).

- (A)  $\text{WD}(V_\lambda)^{\text{Fr-ss}}$  is pure of weight  $w$ ,
- (B)  $\lambda^{\text{intal}} \circ \text{trWD}(\mathcal{V})^{\text{Fr-ss}} = \text{trWD}(V_\lambda)^{\text{Fr-ss}}$ ,
- (C)  $\text{WD}(V_\lambda)^{\text{Fr-ss}}$  is annihilated by the  $D$ -th power of its monodromy where  $D$  denotes the size of  $\text{WD}(\mathcal{V})^{\text{Fr-ss}}$ .

The indecomposable summands of  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  are of size (see definition 2.6) at most  $t_m$  by condition (C) and are of weight  $w$  by condition (A) and remark 2.11. Since the elements of  $S$  are irreducible Frobenius-semisimple  $W_K$ -representations (by lemma 2.9) and the sum of their traces is equal to  $\mathrm{tr}\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  (by condition (B)), the difference of the weights of any two elements of the multiset  $S$  is at most  $2(t_m - 1)$ . Note that the difference of the weights of  $\lambda^{\mathrm{intal}}(r_m)$ ,  $\lambda^{\mathrm{intal}}(|\mathrm{Art}_K^{-1}|_K^{t_m-1}r_m)$  is  $2(t_m - 1)$ . So these are a highest weight and a lowest weight element of  $S$  respectively. By condition (A),  $w$  is equal to the average of the weights of a highest weight and a lowest weight element of  $S$ , *i.e.*, the average of the weights of  $\lambda^{\mathrm{intal}}(r_m)$  and  $\lambda^{\mathrm{intal}}(|\mathrm{Art}_K^{-1}|_K^{t_m-1}r_m)$ . So  $\lambda^{\mathrm{intal}}(r_m)$  has weight  $w + t_m - 1$ . Since  $\lambda^{\mathrm{intal}}(r_m)$  is a highest weight element of  $S$  and  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  is pure of weight  $w$  (by condition (A)), the Weil-Deligne representation  $\mathrm{Sp}_{t_m}(\lambda^{\mathrm{intal}}(r_m))$  is a direct summand of  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$ . Now suppose that for an integer  $1 \leq m' < m$ , the representation  $\mathrm{Sp}_{t_{m'+1}}(\lambda^{\mathrm{intal}} \circ r_{m'+1}) \oplus \cdots \oplus \mathrm{Sp}_{t_m}(\lambda^{\mathrm{intal}} \circ r_m)$  is a direct summand of  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  as Weil-Deligne representations, *i.e.*, there is an isomorphism

$$(4.2) \quad \mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}} \simeq W \oplus \bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\lambda^{\mathrm{intal}} \circ r_i).$$

Let  $\mathcal{W}$  denote the Weil-Deligne representation  $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(r_i)$ . Then the sum  $\sum_{i=m'+1}^m (t_i - t_{m'}) \dim \rho_i$  is equal to the integer  $\dim_{\overline{\mathbb{F}}} N^{t_{m'}}(\mathrm{WD}(\mathcal{V})^{\mathrm{Fr}\text{-ss}})$  (by equation (4.1)), which is larger than  $\dim_{\overline{\mathbb{F}}} \lambda(N)^{t_{m'}}(\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}})$  and this is bigger than  $\dim_{\overline{\mathbb{F}}} \lambda(N)^{t_{m'}} W + \sum_{i=m'+1}^m (t_i - t_{m'}) \dim \rho_i$  (by equation (4.2)). So  $\lambda(N)^{t_{m'}}(W) = 0$ . Thus conditions (A'), (B'), (C') below hold with  $D' = t_{m'}$ .

(A')  $W$  is pure of weight  $w$ ,

(B')  $\lambda^{\mathrm{intal}} \circ \mathrm{tr} \mathcal{W} = \mathrm{tr} W$ ,

(C')  $W$  is annihilated by the  $D'$ -th power of its monodromy where  $D'$  denotes the size of  $\mathcal{W}$ .

Using an argument analogous to the proof of the fact that  $\mathrm{Sp}_{t_m}(\lambda^{\mathrm{intal}} \circ r_m)$  is a direct summand of  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$ , we deduce that the Weil-Deligne representation  $\mathrm{Sp}_{t_{m'}}(\lambda^{\mathrm{intal}} \circ r_{m'})$  is a direct summand of  $W$ . Then equation (4.2) shows that  $\mathrm{Sp}_{t_{m'}}(\lambda^{\mathrm{intal}} \circ r_{m'}) \oplus \mathrm{Sp}_{t_{m'+1}}(\lambda^{\mathrm{intal}} \circ r_{m'+1}) \oplus \cdots \oplus \mathrm{Sp}_{t_m}(\lambda^{\mathrm{intal}} \circ r_m)$  is a direct summand of  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$ . This proves part (1) by induction. Then part (2) to (5) follows.

To simplify notations, we assume that  $\mathcal{K}$  is algebraically closed. Let  $\mathcal{O}_\mu$  (resp.  $\mathcal{O}_\lambda$ ) denote the image of  $\mu$  (resp.  $\lambda$ ) and  $\eta : \mathcal{O}_\mu \rightarrow \mathcal{O}_\lambda$  denote the  $\mathbb{Z}_p$ -algebra homomorphism such that  $\lambda = \eta \circ \mu$ . Let  $\lambda^\dagger$  denote the map  $\eta^{\mathrm{intal}} \circ \mu^{\mathrm{intal}}$ . By proposition 2.8, there exist positive integers  $M, t'_1 \leq \cdots \leq t'_M$  and irreducible Frobenius-semisimple representations  $s_1, \dots, s_M$  over  $\mathcal{O}_\mu^{\mathrm{intal}}[1/p]$  such that  $\mathrm{WD}(V_\mu)^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\bigoplus_{i=1}^M \mathrm{Sp}_{t'_i}(s_i)$ . By part (1),  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\bigoplus_{i=1}^M \mathrm{Sp}_{t'_i}(\eta^{\mathrm{intal}} \circ s_i)$ . Hence  $M = m$  and  $t'_i = t_i$  for all  $1 \leq i \leq M$ . So  $\eta^{\mathrm{intal}} \circ s_i, \lambda^\dagger \circ r_i$  are of weight  $w + t_i - 1$  for all  $1 \leq i \leq m$ . Note that for some integer  $1 \leq j \leq m$  and  $0 \leq a \leq t_j - 1$ , the representations  $\mu^{\mathrm{intal}} \circ r_m$  and  $s_j |\mathrm{Art}_K^{-1}|_K^a$  are isomorphic. So the representations  $\lambda^\dagger \circ r_m, \eta^{\mathrm{intal}} \circ (s_j |\mathrm{Art}_K^{-1}|_K^a)$  are of equal weight. This shows  $t_m = t_j - 2a$  and hence  $a = 0, t_j = t_m$ . Thus  $\mathrm{Sp}_{t_m}(\mu^{\mathrm{intal}} \circ r_m)$  is a direct summand of  $\mathrm{WD}(V_\mu)^{\mathrm{Fr}\text{-ss}}$ . Now suppose that for an integer  $1 \leq m' < m$ , the representation  $\bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$  is a direct summand of  $\mathrm{WD}(V_\mu)^{\mathrm{Fr}\text{-ss}}$ . So by proposition 2.8, there exist irreducible Frobenius-semisimple representations  $s'_1, \dots, s'_{m'}$  over  $\mathcal{O}_\mu^{\mathrm{intal}}[1/p]$  such that

$\mathrm{WD}(V_\mu)^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(s'_i) \oplus \bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$ . By part (1),  $\mathrm{WD}(V_\lambda)^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(\eta^{\mathrm{intal}} \circ s'_i) \oplus \bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\eta^{\mathrm{intal}} \circ \mu^{\mathrm{intal}} \circ r_i)$ . So  $\eta^{\mathrm{intal}} \circ s'_i, \lambda^\dagger \circ r_i$  are of weight  $w + t_i - 1$  for all  $1 \leq i \leq m'$ . Note that for some integer  $1 \leq k \leq m'$  and  $0 \leq b \leq t_k - 1$ , the representations  $\mu^{\mathrm{intal}} \circ r_{m'}$  and  $s'_k | \mathrm{Art}_K^{-1} |_K^b$  are isomorphic. So the representations  $\lambda^\dagger \circ r_{m'}, \eta^{\mathrm{intal}} \circ (s'_k | \mathrm{Art}_K^{-1} |_K^b)$  are of equal weight. This shows  $t_{m'} = t_k - 2b$  and hence  $b = 0, t_k = t_{m'}$ . Thus  $\mathrm{Sp}_{t_{m'}}(\mu^{\mathrm{intal}} \circ r_{m'})$  is a direct summand of  $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(s'_i)$  and hence  $\bigoplus_{i=m'}^m \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$  is a direct summand of  $\mathrm{WD}(V_\mu)^{\mathrm{Fr}\text{-ss}}$ . This completes the proof by induction.  $\square$

## 5. PURITY FOR PSEUDOREPRESENTATIONS

Let  $\mathcal{O}$  be an integral domain containing  $\mathbb{Z}_p$  as a subalgebra. We denote its fraction field by  $\mathcal{L}$ .

**5.1. Preliminaries.** Let  $\mathcal{O}_1, \mathcal{O}_2$  be integral domains containing  $\mathbb{Z}_p$  as a subalgebra. We denote their fraction fields by  $\mathcal{L}_1, \mathcal{L}_2$  respectively. Let  $\mathrm{res}_1 : \mathcal{O} \hookrightarrow \mathcal{O}_1, \mathrm{res}_2 : \mathcal{O} \hookrightarrow \mathcal{O}_2$  be injective  $\mathbb{Z}_p$ -algebra homomorphisms. Let  $T_0 : W_K \rightarrow \mathcal{O}^{\mathrm{intal}}[1/p]$  be a pseudorepresentation of dimension  $d \geq 1$  and  $(r_1, N_1) : W_K \rightarrow \mathrm{GL}_d(\mathcal{O}_1^{\mathrm{intal}}[1/p]), (r_2, N_2) : W_K \rightarrow \mathrm{GL}_d(\mathcal{O}_2^{\mathrm{intal}}[1/p])$  be Weil-Deligne representations such that

$$(5.1) \quad \mathrm{res}_1^{\mathrm{intal}} \circ T_0 = \mathrm{tr}(r_1), \quad \mathrm{res}_2^{\mathrm{intal}} \circ T_0 = \mathrm{tr}(r_2).$$

Suppose that there exist  $\mathbb{Z}_p$ -algebra homomorphisms  $f_1 : \mathcal{O}_1^{\mathrm{intal}} \rightarrow \overline{\mathbb{Q}_p}, f_2 : \mathcal{O}_2^{\mathrm{intal}} \rightarrow \overline{\mathbb{Q}_p}$  such that  $f_1 \circ (r_1, N_1), f_2 \circ (r_2, N_2)$  are pure. We first state two propositions. Then we prove a lemma which will be used to establish these propositions. For the notion of size, we refer to definition 2.6.

**Proposition 5.1.** *The size of  $(f_1 \circ (r_1, N_1))^{\mathrm{Fr}\text{-ss}}$  is smaller than the size of  $(f_2 \circ (r_2, N_2))^{\mathrm{Fr}\text{-ss}}$ . Consequently, these two representations have the same size.*

Let  $\kappa, t_1 \leq \dots \leq t_\kappa$  be positive integers and  $\theta_{11}, \dots, \theta_{1t_1}, \theta_{21}, \dots, \theta_{2t_2}, \dots, \theta_{\kappa 1}, \dots, \theta_{\kappa t_\kappa}$  be irreducible Frobenius-semisimple representations of  $W_K$  over  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that

- (1)  $T_0$  is equal to  $\sum_{i=1}^\kappa \sum_{j=1}^{t_i} \mathrm{tr} \theta_{ij}$ ,
- (2) for any  $1 \leq i \leq \kappa, 1 \leq j \leq t_i$ , the representations  $\mathrm{res}_1^{\mathrm{intal}} \circ \theta_{ij}, \mathrm{res}_1^{\mathrm{intal}} \circ (|\mathrm{Art}_K^{-1} |_K^{j-1} \theta_{i1})$  of  $W_K$  are isomorphic over  $\overline{\mathcal{L}_1}$  and
- (3) there is an isomorphism

$$(5.2) \quad ((r_1, N_1) \otimes_{\mathcal{O}_1} \overline{\mathcal{L}_1})^{\mathrm{Fr}\text{-ss}} \simeq \bigoplus_{i=1}^\kappa \mathrm{Sp}_{t_i}(\mathrm{res}_1^{\mathrm{intal}} \circ \theta_{i1}).$$

**Proposition 5.2.** *The representation  $\mathrm{Sp}_{t_\kappa}(\mathrm{res}_2^{\mathrm{intal}} \circ \theta_{\kappa 1})$  is a direct summand of  $((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}_2})^{\mathrm{Fr}\text{-ss}}$  as Weil-Deligne representations.*

**Lemma 5.3.** *Let  $k, s_1 \leq \dots \leq s_k$  be positive integers and  $\vartheta_1, \dots, \vartheta_k$  be irreducible Frobenius-semisimple representations of  $W_K$  over  $\mathcal{O}_2^{\mathrm{intal}}[1/p]$  such that*

$$(5.3) \quad ((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}_2})^{\mathrm{Fr}\text{-ss}} \simeq \bigoplus_{i=1}^k \mathrm{Sp}_{s_i}(\vartheta_i).$$

Then for some integers  $1 \leq a, b \leq k$ , we have

$$(5.4) \quad (\text{res}_2^{\text{intal}} \circ \theta_{\kappa t_\kappa})|_{\overline{\mathcal{L}}_2} \simeq (\vartheta_a |\text{Art}_K^{-1}|_K^{s_a-1})|_{\overline{\mathcal{L}}_2}, \quad (\text{res}_2^{\text{intal}} \circ \theta_{\kappa 1})|_{\overline{\mathcal{L}}_2} \simeq (\vartheta_b)|_{\overline{\mathcal{L}}_2},$$

$$(5.5) \quad 2t_\kappa = s_a + s_b \leq 2s_k.$$

*Proof.* By lemma 2.9,  $\text{res}_1^{\text{intal}} \circ \theta_{i1}$  is an irreducible Frobenius-semisimple representation of  $W_K$  over  $\mathcal{O}_1^{\text{intal}}[1/p]$ . Since  $f_1 \circ (r_1, N_1)$  is pure, theorem 4.1 and equation (5.2) give

$$(5.6) \quad (f_1 \circ (r_1, N_1))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{\kappa} \text{Sp}_{t_i}(f_1 \circ \text{res}_1^{\text{intal}} \circ \theta_{i1}).$$

Since  $t_1 \leq \dots \leq t_\kappa$  and  $f_1 \circ (r_1, N_1)$  is pure, by equation (5.6), no eigenvalue of  $\phi$  on  $f_1 \circ (r_1, N_1)$  has weight strictly more (resp. less) than the weight of the  $\phi$ -eigenvalues on  $\theta_{\kappa 1}$  (resp.  $\theta_{\kappa t_\kappa}$ ). So there are no integers  $i, j$  with  $1 \leq i \leq \kappa, 1 \leq j \leq t_i$  such that  $\theta_{ij}$  is isomorphic to  $\theta_{\kappa 1} |\text{Art}_K^{-1}|_K^{-\nu}$  or  $\theta_{\kappa t_\kappa} |\text{Art}_K^{-1}|_K^\nu$  for some integer  $\nu \geq 1$ . Note that by lemma 2.9, there exist integers  $1 \leq a, b \leq k$  such that the  $W_K$ -representation  $\text{res}_2^{\text{intal}} \circ \theta_{\kappa \alpha_\kappa}$  (resp.  $\text{res}_2^{\text{intal}} \circ \theta_{\kappa 1}$ ) is isomorphic to  $\vartheta_a |\text{Art}_K^{-1}|_K^{j_1}$  (resp.  $\vartheta_b |\text{Art}_K^{-1}|_K^{j_2}$ ) over  $\overline{\mathcal{L}}_2$  where  $0 \leq j_1 \leq s_a - 1$  (resp.  $0 \leq j_2 \leq s_b - 1$ ). Now for some  $1 \leq i \leq \kappa, 1 \leq j \leq t_i$ , the  $W_K$ -representations  $\text{res}_2^{\text{intal}} \circ \theta_{ij}, \vartheta_a |\text{Art}_K^{-1}|_K^{s_a-1} = (\text{res}_2^{\text{intal}} \circ \theta_{\kappa \alpha_\kappa}) |\text{Art}_K^{-1}|_K^{s_a-1-j_1}$  are isomorphic over  $\overline{\mathcal{L}}_2$ . As  $\text{res}_2$  is injective and the traces of the representations  $\theta_{ij}$  and  $|\text{Art}_K^{-1}|_K^{s_a-1-j_1} \theta_{\kappa \alpha_\kappa}$  coincide after composing them with  $\text{res}_2^{\text{intal}}$ , these representations are isomorphic over  $\overline{\mathcal{L}}$  (by [Ser98, Chapter 1, §2] for instance). As noted before,  $s_a - 1 - j_1$  cannot be positive. So  $j_1$  is equal to  $s_a - 1$ . Similarly  $j_2$  is zero. Thus equation (5.4) holds.

Let  $w$  denote the weight of the pure representation  $f_2 \circ (r_2, N_2)$ . By theorem 4.1 and equation (5.3),

$$(5.7) \quad (f_2 \circ (r_2, N_2))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^k \text{Sp}_{s_i}(f_2 \circ \vartheta_i).$$

So the weight of any  $\phi$ -eigenvalue on  $f_2 \circ \vartheta_b$  (resp.  $f_2 \circ \vartheta_a |\text{Art}_K^{-1}|_K^{s_a-1}$ ) is equal to  $w + (s_b - 1)$  (resp.  $w - (s_a - 1)$ ). So their difference, denoted  $\delta$ , is equal to  $s_a + s_b - 2$ . On the other hand, since  $\theta_{\kappa t_\kappa}$  and  $|\text{Art}_K^{-1}|_K^{t_\kappa-1} \theta_{\kappa 1}$  are isomorphic over  $\overline{\mathcal{L}}$  (as their traces become equal after composing them with  $\text{res}_1^{\text{intal}}$  and  $\text{res}_1$  is injective), by equation (5.4),  $\delta$  is equal to  $2(t_\kappa - 1)$ . Since  $s_a, s_b$  are smaller than  $s_k$ , we get equation (5.5).  $\square$

*Proof of proposition 5.1.* Equation (5.5), (5.6), (5.7) give the first part of proposition 5.1. Then the second part follows.  $\square$

*Proof of proposition 5.2.* By proposition 5.1,  $t_\kappa$  is equal to  $s_k$ . Then equation (5.5) gives  $s_a = s_b = s_k$ . So  $\text{Sp}_{s_b}(\vartheta_b) = \text{Sp}_{s_k}(\text{res}_2^{\text{intal}} \circ \theta_{\kappa 1})$  is a direct summand of  $((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}}_2)^{\text{Fr-ss}}$ .  $\square$

**5.2. Pseudorepresentations of Weil groups.** Let  $A$  be a commutative ring and  $R$  be an  $A$ -algebra. Given a pseudorepresentation  $T : R \rightarrow A$  of dimension  $d \geq 1$ , the degree  $d$  monic polynomial  $P_{x,T}(X) = X^d + (-1)^{d-1} T(x) X^{d-1} + \dots$  (as defined in [BC09, §1.2.3]) is called the characteristic polynomial of  $x$  (for  $T$ ). It has coefficients in  $A[1/d!]$ .

**Theorem 5.4** (Purity for pseudorepresentations). *Let  $\mathcal{O}$  be an integral domain over  $\mathbb{Z}_p$  and  $\text{res} : \mathcal{O} \hookrightarrow \mathcal{O}$  be an injective  $\mathbb{Z}_p$ -algebra homomorphism. Let  $T : W_K \rightarrow \mathcal{O}$  be a*

pseudorepresentation of dimension  $n \geq 1$  and let  $(r, N) : W_K \rightarrow \mathrm{GL}_n(\mathcal{O}[1/p])$  be a Weil-Deligne representation such that  $\mathrm{res} \circ T = \mathrm{trr}$ . Suppose  $f \circ (r, N)$  is pure for some  $\mathbb{Z}_p$ -algebra homomorphism  $f : \mathcal{O} \rightarrow \overline{\mathbb{Q}_p}$ . Then there exist positive integers  $m, t_1 \leq t_2 \leq \dots \leq t_m$  and irreducible Frobenius-semisimple representations  $r_1, \dots, r_m$  of  $W_K$  with coefficients in  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that the statements (1), (2), (3) hold.

(1)  $T$  is equal to  $\sum_{i=1}^m \sum_{j=1}^{t_i} \mathrm{trr}_i |\mathrm{Art}_K^{-1}|_K^{j-1}$ .

(2) If there exist an integral domain  $\mathcal{O}'$  over  $\mathbb{Z}_p$  and a Weil-Deligne representation  $(r', N') : W_K \rightarrow \mathrm{GL}_n(\mathcal{O}'[1/p])$  such that

- $\mathrm{res}' \circ T = \mathrm{trr}'$  for some injective  $\mathbb{Z}_p$ -algebra homomorphism  $\mathrm{res}' : \mathcal{O} \hookrightarrow \mathcal{O}'$ ,
- $f' \circ (r', N')$  is pure for some  $\mathbb{Z}_p$ -algebra homomorphism  $f' : \mathcal{O}' \rightarrow \overline{\mathbb{Q}_p}$ ,

then for any lift  $\mathrm{res}'^\dagger$  of  $\mathrm{res}'$  and any lift  $f'^\dagger$  of  $f'$ , there are isomorphisms

$$(5.8) \quad ((r', N') \otimes_{\mathcal{O}'} \overline{Q}(\mathcal{O}'))^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\mathrm{res}'^\dagger \circ r_i),$$

$$(5.9) \quad (f' \circ (r', N'))^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(f'^\dagger \circ \mathrm{res}'^\dagger \circ r_i).$$

(3) If the characteristic polynomial  $P_{\phi, T}(X)$  of  $\phi$  has coefficients in  $\mathcal{O}^{\mathrm{intal}} \cap \mathcal{O}[1/n!]$ , then  $r_i$  has values in  $\mathcal{O}^{\mathrm{intal}}$  whenever  $r_i$  is a character for some  $1 \leq i \leq m$ .

Moreover, if there are positive integers  $M, s_1, \dots, s_M$  and irreducible Frobenius-semisimple representations  $R_1, \dots, R_M$  of  $W_K$  over  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that the statements (1), (2) above hold (when  $m, t_i, r_i$  are replaced by  $M, s_i, R_i$  respectively), then  $m = M, t_1 = s_1, \dots, t_m = s_M$  and there exists a permutation  $\sigma$  on  $\{1, \dots, m\}$  such that

- (i)  $r_{\sigma(i)}$  is isomorphic to  $R_i$  over  $\overline{\mathcal{L}}$  for all  $1 \leq i \leq m$ ,
- (ii)  $\{a, a+1, \dots, b\}$  is stable under the action of  $\sigma$  whenever  $t_{a-1} < t_a = \dots = t_b < t_{b+1}$  for some integers  $1 \leq a, b \leq m$  (here  $t_0 := 0, t_{m+1} := t_m + 1$ ).

*Proof.* Let  $\mathcal{L}$  denote the fraction field of  $\mathcal{O}$ . By proposition 2.8, there exist positive integers  $m, t_1 \leq t_2 \leq \dots \leq t_m$  and irreducible Frobenius-semisimple representations  $\tau_1, \dots, \tau_m$  of  $W_K$  with coefficients in  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that  $((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^{\mathrm{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\tau_i)$ . Since  $\mathrm{trr} = \mathrm{res} \circ T$ , the characteristic polynomial of  $\tau_i$  has coefficients in  $(\mathrm{res}\mathcal{O})^{\mathrm{intal}}[1/p]$  (we consider  $Q(\mathrm{res}\mathcal{O})$  as a subfield of  $\mathcal{L}$  and thus  $(\mathrm{res}\mathcal{O})^{\mathrm{intal}}$  is a subring of  $\mathcal{O}^{\mathrm{intal}}$ ). So by proposition 2.8, we may (and do) assume that  $\tau_i$  has coefficients in  $(\mathrm{res}\mathcal{O})^{\mathrm{intal}}[1/p]$ . So there exist irreducible Frobenius-semisimple representations  $r_1, \dots, r_m$  of  $W_K$  with coefficients in  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that  $\mathrm{res} \circ r_1 = \tau_1, \dots, \mathrm{res} \circ r_m = \tau_m$ . Since  $T - \sum_{i=1}^m \sum_{j=1}^{t_i} \mathrm{trr}_i |\mathrm{Art}_K^{-1}|_K^{j-1}$  goes to zero under  $\mathrm{res}^{\mathrm{intal}}$  and  $\mathrm{res}$  is injective, we get part (1).

Let  $\mathcal{L}'$  denote the fraction field of  $\mathcal{O}'$ . By proposition 5.2,  $\mathrm{Sp}_{t_m}(\mathrm{res}'^\dagger \circ r_m)$  is a direct summand of  $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\mathrm{Fr-ss}}$ . Suppose for some  $1 \leq k < m$ ,  $\bigoplus_{i=k+1}^m \mathrm{Sp}_{t_i}(\mathrm{res}'^\dagger \circ r_i)$  is a direct summand of  $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\mathrm{Fr-ss}}$ . We will now show that  $\bigoplus_{i=k}^m \mathrm{Sp}_{t_i}(\mathrm{res}'^\dagger \circ r_i)$  is a direct summand of  $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\mathrm{Fr-ss}}$ . By proposition 2.8, there exist positive integers  $Q, s_1 \leq \dots \leq s_Q$  and irreducible Frobenius-semisimple representations  $\eta_1, \dots, \eta_Q$  of  $W_K$  with coefficients in  $\mathcal{O}^{\mathrm{intal}}[1/p]$  such that  $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\mathrm{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^Q \mathrm{Sp}_{s_i}(\eta_i) \oplus \bigoplus_{i=k+1}^m \mathrm{Sp}_{t_i}(\mathrm{res}'^\dagger \circ r_i)$ . Note that the specialization of the pseudorepresentation  $\sum_{i=1}^k \sum_{j=1}^{t_i} \mathrm{trr}_i |\mathrm{Art}_K^{-1}|_K^{j-1} :$



$W_K \rightarrow \mathcal{O}^{\text{intal}}[1/p]$  under  $\text{res}^{\text{intal}}$  (resp.  $\text{res}^{\dagger}$ ) is equal to the trace of the Weil-Deligne representation  $\oplus_{i=1}^k \text{Sp}_{t_i}(\tau_i)$  (resp.  $\oplus_{i=1}^Q \text{Sp}_{s_i}(\eta_i)$ ) of  $W_K$  with coefficients in  $\mathcal{O}^{\text{intal}}[1/p]$  (resp.  $\mathcal{O}^{\text{intal}}[1/p]$ ). So by proposition 5.2, the representation  $\text{Sp}_{t_k}(\text{res}^{\dagger} \circ r_k)$  is a direct summand of  $\oplus_{i=1}^Q \text{Sp}_{s_i}(\eta_i)$ . This shows that  $\oplus_{i=k}^m \text{Sp}_{t_i}(\text{res}^{\dagger} \circ r_i)$  is a direct summand of  $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\text{Fr-ss}}$ . So we obtain equation (5.8) by induction. Using theorem 4.1, we get equation (5.9). Part (3) is clear.

To establish the final part, note that  $((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^{\text{Fr-ss}}$  is isomorphic to  $\oplus_{i=1}^m \text{Sp}_{t_i}(\text{res}^{\text{intal}} \circ r_i)$  and  $\oplus_{i=1}^M \text{Sp}_{s_i}(\text{res}^{\text{intal}} \circ R_i)$ . This shows that  $m = M, t_1 = s_1, \dots, t_m = s_M$ . By theorem 2.5, there exists a permutation  $\sigma$  on  $\{1, \dots, m\}$  such that condition (ii) above holds and  $\text{res}^{\text{intal}} \circ r_{\sigma(i)}$  is isomorphic to  $\text{res}^{\text{intal}} \circ R_i$  for all  $1 \leq i \leq m$ . Since  $\text{res}$  is injective,  $\text{res}^{\text{intal}}$  is also injective. So  $r_{\sigma(i)}$  and  $R_i$  have same traces and hence these are isomorphic over  $\overline{\mathcal{L}}$  (by [Ser98, Chapter 1, §2] for instance).  $\square$

**5.3. Pure specializations of pseudorepresentations of global Galois groups.** Given a local ring  $(A, \mathfrak{m})$ , we denote its Henselization by  $(A^h, \mathfrak{m}^h)$  (see [Sta14, Tag 04GQ]) and consider their residue fields to be equal via the isomorphism  $A/\mathfrak{m} \rightarrow A^h/\mathfrak{m}^h$  (see [Sta14, Tag 04GN]). Since the map  $A \rightarrow A^h$  is flat (by [Sta14, Tag 07QM] for instance) and flat maps satisfy going down property (see [Sta14, Tag 00HS]), the minimal primes of  $A^h$  go to the minimal primes of  $A$  under the inverse of the map  $A \rightarrow A^h$ . Given a prime ideal  $\mathfrak{p}$  of a ring  $R$ , the mod  $\mathfrak{p}$  reduction map is denoted by  $\pi_{\mathfrak{p}}$ .

Let  $F$  be a number field and  $T : G_F \rightarrow \mathcal{O}$  be a pseudorepresentation such that  $T = T_1 + \dots + T_n$  where  $T_1 : G_F \rightarrow \mathcal{O}, \dots, T_n : G_F \rightarrow \mathcal{O}$  are traces of irreducible representations  $\sigma_1, \dots, \sigma_n$  of  $G_F$  over  $\overline{\mathcal{L}}$ . Let  $w \nmid p$  be a finite place of  $F$  and assume that the restrictions of  $\sigma_1, \dots, \sigma_n$  to  $W_w$  are monodromic.

**Definition 5.5.** *The irreducibility and  $w$ -purity locus (irreducibility and purity locus, in short) of  $T_1, \dots, T_n$  is defined to be the collection of all tuples of the form  $(\mathcal{O}, \mathfrak{p}, \kappa, \text{res}, \rho_1, \dots, \rho_n)$  where  $\mathcal{O}$  is a domain over  $\mathbb{Z}_p$ ,  $\mathfrak{p}$  is a prime ideal of  $\mathcal{O}$  such that the Henselization  $\mathcal{O}_{\mathfrak{p}}^h$  of  $\mathcal{O}_{\mathfrak{p}}$  is Hausdorff,  $\kappa$  denotes the residue field  $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$  and is an algebraic extension of  $\mathbb{Q}_p$ ,  $\text{res} : \mathcal{O} \hookrightarrow \mathcal{O}$  is an injective  $\mathbb{Z}_p$ -algebra homomorphism and for each  $1 \leq i \leq n$ ,  $\rho_i$  is an irreducible  $G_F$ -representation over  $\overline{\kappa}$  such that  $\pi_{\mathfrak{p}} \circ \text{res} \circ T_i$  is equal to the trace of  $\rho_i$  and  $\rho_i|_{G_w}$  is pure (of some weight depending on  $i$ ).*

For each element  $(\mathcal{O}, \mathfrak{p}, \kappa, \text{res}, \rho_1, \dots, \rho_n)$  of this locus, we choose semisimple  $G_F$ -representations  $\tilde{\rho}_1, \dots, \tilde{\rho}_n$  over  $\overline{\mathbb{Q}}(\mathcal{O})$  such that  $\text{tr} \tilde{\rho}_i = \text{res} \circ T_i$  for all  $1 \leq i \leq n$  (using [Tay91, Theorem 1]) and choose  $G_F$ -representations  $\varrho_1, \dots, \varrho_n$  over  $\mathcal{O}_{\mathfrak{p}}^h$  such that  $\text{tr} \varrho_i = \text{res} \circ T_i$  for all  $1 \leq i \leq n$  (using [Nys96, Théorème 1] and the fact that  $\mathcal{O}_{\mathfrak{p}}^h$  is Hausdorff). We also fix a minimal prime  $\mathfrak{a}$  of  $\mathcal{O}_{\mathfrak{p}}^h$ . The composite maps  $\mathcal{O} \rightarrow \mathcal{O}/\mathfrak{p} \rightarrow \kappa$ ,  $\mathcal{O} \rightarrow \mathcal{O}_{\mathfrak{p}} \rightarrow \mathcal{O}_{\mathfrak{p}}^h$  and  $\mathcal{O} \rightarrow \mathcal{O}_{\mathfrak{p}} \rightarrow \mathcal{O}_{\mathfrak{p}}^h \rightarrow \mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a}$  are denoted by  $\pi_{\mathfrak{p}}$ ,  $h_{\mathfrak{p}}$  and  $\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}}$  respectively. Note that the map  $\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}}$  is injective (as observed in the beginning of §5.3).

**Theorem 5.6.** *Suppose that the irreducibility and purity locus of  $T_1, \dots, T_n$  is nonempty. Then there exist positive integers  $m, t_1 \leq t_2 \leq \dots \leq t_m$  and irreducible Frobenius-semisimple representations  $r_1, \dots, r_m$  of  $W_w$  with coefficients in  $\mathcal{O}^{\text{intal}}[1/p]$  such that the following hold.*

- (1)  $T|_{W_w}$  is equal to  $\sum_{i=1}^m \sum_{j=1}^{t_i} \text{trr}_i | \text{Art}_K^{-1} |_K^{j-1}$ .
- (2) If  $(\mathcal{O}, \mathfrak{p}, \kappa, \text{res}, \rho_1, \dots, \rho_n)$  is an element of the irreducibility and purity locus of  $T_1, \dots, T_n$ , then for any lift  $\text{res}^{\dagger}$  (resp.  $\pi_{\mathfrak{p}}^{\dagger}, (\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}})^{\dagger}$ ) of  $\text{res}$  (resp.  $\pi_{\mathfrak{p}}, \pi_{\mathfrak{a}} \circ h_{\mathfrak{p}}$ ), there are

isomorphisms

$$(5.10) \quad \mathrm{WD} \left( \bigoplus_{i=1}^n \rho_i|_{W_w} \right)^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\pi_{\mathfrak{p}}^{\dagger} \circ \mathrm{res}^{\dagger} \circ r_i),$$

$$(5.11) \quad \mathrm{WD} \left( \bigoplus_{i=1}^n (\pi_{\mathfrak{a}} \circ \varrho_i) \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})|_{W_w} \right)^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}((\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}})^{\dagger} \circ \mathrm{res}^{\dagger} \circ r_i),$$

$$(5.12) \quad \mathrm{WD} \left( \bigoplus_{i=1}^n \tilde{\rho}_i|_{W_w} \right)^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\mathrm{res}^{\dagger} \circ r_i),$$

$$(5.13) \quad \mathrm{WD} \left( \bigoplus_{i=1}^n \sigma_i|_{W_w} \right)^{\mathrm{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i).$$

*Proof.* Since  $(\pi_{\mathfrak{a}} \circ \varrho_i) \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$  is irreducible and the  $G_F$ -representations  $\tilde{\rho}_i \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$ ,  $(\pi_{\mathfrak{a}} \circ \varrho_i) \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$  have same traces, these are isomorphic. Similarly  $\sigma_i \otimes \overline{Q}(\mathcal{O})$  and  $\tilde{\rho}_i$  are isomorphic. Since  $\sigma_i|_{W_w}$  is monodromic,  $\pi_{\mathfrak{a}} \circ \varrho_i|_{W_w}$  is monodromic. Note that  $\mathrm{WD}(\pi_{\mathfrak{a}} \circ \varrho_i|_{W_w})$  has coefficients in  $(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})[1/p]$ , its trace is equal to  $\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}} \circ \mathrm{res} \circ T|_{W_w}$  and it has a pure specialization  $\mathrm{WD}(\rho_i|_{W_w})$ . Also note that the map  $\pi_{\mathfrak{a}} \circ h_{\mathfrak{p}} \circ \mathrm{res}$  is an injective  $\mathbb{Z}_p$ -algebra homomorphism from  $\mathcal{O}$  to the domain  $\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a}$  over  $\mathbb{Z}_p$ . Then theorem 5.4 gives part (1) and equation (5.10), (5.11). Since  $(\pi_{\mathfrak{a}} \circ \varrho_i) \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$  is isomorphic to  $\tilde{\rho}_i \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$  and  $\sigma_i \otimes \overline{Q}(\mathcal{O}_{\mathfrak{p}}^h/\mathfrak{a})$ , we get equation (5.12) and (5.13) from equation (5.11).  $\square$

## 6. LOCAL LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}_n$ IN FAMILIES

In this section, we use theorem 4.1 to strenthen a part of the local Langlands correspondence for  $\mathrm{GL}_n$  in families (see theorem 3.2) formulated by Emerton and Helm. Let  $S, G, r_v$  be as in theorem 3.1 and suppose that the  $A[G]$ -module  $\tilde{\pi}(\{r_v\}_{v \in S})$  exists.

**Theorem 6.1.** *Let  $\mathfrak{p}$  be a prime of  $A[1/p]$ . Suppose there exists a  $\mathbb{Z}_p$ -algebra homomorphism  $i_{\mathfrak{p}} : A \rightarrow \overline{\mathbb{Q}}_p$  such that  $\mathfrak{p}$  is contained inside the kernel of  $i_{\mathfrak{p}}$  and  $r_v \otimes_{A, i_{\mathfrak{p}}} \overline{\mathbb{Q}}_p$  is pure for all  $v \in S$ . Then the surjection  $\varsigma_{\mathfrak{p}}$  as in theorem 3.2 is an isomorphism. If  $\mathfrak{p}$  lies on only one irreducible component of  $\mathrm{Spec} A[1/p]$ , then the surjection  $\gamma_{\mathfrak{p}}$  as in theorem 3.2 is also an isomorphism.*

*Proof.* Let  $\mathfrak{a}$  denote a minimal prime of  $A$  contained in  $\mathfrak{p}$ . Then by theorem 4.1, the rank of the  $i$ -th power of monodromy of  $r_{v, \mathfrak{a}}$  is equal to the rank of the  $i$ -th power of the monodromy of  $r_{v, \mathfrak{p}}$  for any  $i \geq 1$  and any  $v \in S$ . Hence the result follows from theorem 3.2.  $\square$

## 7. FAMILIES OF GALOIS REPRESENTATIONS

The goal of this section is to illustrate the role of purity for big Galois representations, purity for pseudorepresentations and theorem 5.6 in the study of variation of local Euler factors, local automorphic types, intersection points of irreducible components etc. for families of Galois representations.

**7.1. Hida families.** For Hida theory of ordinary cusp forms, we follow [Hid87] and refer to the references [Hid86a, Hid86b] contained therein. We follow [Ger10] for Hida theory for definite unitary groups.

7.1.1. *Cusp forms.* Let  $f = \sum_{n=1}^{\infty} a_n(f)q^n$  be a normalized eigen cusp form of weight  $k \geq 2$ . Then by [Eic54, Shi58] (for  $k = 2$ ), [Del69] (for  $k > 2$ ), there exists a unique (up to equivalence) continuous Galois representation  $\rho_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  such that  $\mathrm{tr} \rho_f(\mathrm{Fr}_{\ell}) = a_{\ell}(f)$  for any prime  $\ell$  not dividing  $p$  and the level of  $f$ . Let  $\pi(f) = \otimes'_{\ell \leq \infty} \pi(f)_{\ell}$  denote the irreducible unitary representation of  $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$  corresponding to  $f$  (see [Gel75, Theorem 5.19, 4.30]).

Let  $N$  be a positive integer and  $p$  be an odd prime with  $p \nmid N$  and  $Np \geq 4$ . Let  $h^{\mathrm{ord}}$  be the universal  $p$ -ordinary Hecke algebra of tame level  $N$  (denoted  $h^{\mathrm{ord}}(N; \mathbb{Z}_p)$  in [Hid87]). It has an algebra structure over  $\mathbb{Z}_p[[X]]$ . Let  $\mathfrak{a}$  be a minimal prime of  $h^{\mathrm{ord}}$ . Let  $\mathcal{R}(\mathfrak{a})$  denote the ring  $h^{\mathrm{ord}}/\mathfrak{a}$  and  $\mathcal{Q}(\mathfrak{a})$  denote the fraction field of  $\mathcal{R}(\mathfrak{a})$ . Let  $\overline{\mathcal{Q}}(\mathfrak{a})$  be an algebraic closure of  $\mathcal{Q}(\mathfrak{a})$ . Let  $S$  denote the set of places of  $\mathbb{Q}$  dividing  $Np\infty$ . By [Hid87, Theorem 3.1], there exists a unique (up to equivalence) continuous (in the sense of [Hid87, §3]) Galois representation  $\rho_{\mathfrak{a}} : G_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_2(\mathcal{Q}(\mathfrak{a}))$  such that  $\rho_{\mathfrak{a}}$  has traces in  $\mathcal{R}(\mathfrak{a})$  and  $\mathrm{tr}(\rho_{\mathfrak{a}}(\mathrm{Fr}_{\ell})) = T_{\ell} \bmod \mathfrak{a}$  for all prime  $\ell \nmid Np$  where  $T_{\ell} \in h^{\mathrm{ord}}$  denotes the Hecke operator associated to  $\ell$ . Henceforth, the representation  $\rho_{\mathfrak{a}} \otimes \overline{\mathcal{Q}}(\mathfrak{a})$  is denoted by  $\rho_{\mathfrak{a}}$ . A  $\mathbb{Z}_p$ -algebra homomorphism  $\lambda : h^{\mathrm{ord}} \rightarrow \overline{\mathbb{Q}}_p$  is said to be an *arithmetic specialization* if  $\lambda((1+X)^{p^r} - (1+p)^{kp^r}) = 0$  for some integers  $k \geq 0$  and  $r \geq 0$ . The arithmetic specializations of  $h^{\mathrm{ord}}$  are in one-to-one correspondence (by the isomorphism of [Hid87, Theorem 2.2]) with the  $p$ -ordinary  $p$ -stabilized (in the sense of [Wil88, p. 538]) normalized eigen cusp forms of tame level a divisor of  $N$ . Given an arithmetic specialization  $\lambda$  of  $h^{\mathrm{ord}}$ , let  $f_{\lambda}$  denote the corresponding ordinary form.

**Definition 7.1.** *The automorphic type of a minimal prime  $\mathfrak{a}$  of  $h^{\mathrm{ord}}$  at a prime  $\ell \neq p$  is defined to be the unordered tuple  $\mathrm{AT}_{\ell}(\mathfrak{a})$  if the automorphic types of  $\pi(f_{\lambda})_{\ell}$  are equal to  $\mathrm{AT}_{\ell}(\mathfrak{a})$  for all arithmetic specialization  $\lambda$  of  $h^{\mathrm{ord}}$  with  $\lambda(\mathfrak{a}) = 0$ .*

**Theorem 7.2.** *Let  $\mathfrak{a}$  be a minimal prime of  $h^{\mathrm{ord}}$  and  $\ell \neq p$  be a prime. Then the following hold.*

- (1) *If  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is indecomposable and irreducible, then there exists an irreducible Frobenius-semisimple representation  $r$  over  $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}[1/p]$  such that  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $r \otimes \overline{\mathcal{Q}}(\mathfrak{a})$  and  $\mathrm{WD}(\rho_{f_{\lambda}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\lambda^{\mathrm{intal}} \circ r$  for any arithmetic specialization  $\lambda$  of  $h^{\mathrm{ord}}$  with  $\lambda(\mathfrak{a}) = 0$ .*
- (2) *If  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is indecomposable and reducible, then there exists an  $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}$ -valued character  $\chi$  of  $W_{\ell}$  such that  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\mathrm{Sp}_2(\chi) \otimes \overline{\mathcal{Q}}(\mathfrak{a})$  and  $\mathrm{WD}(\rho_{f_{\lambda}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\lambda^{\mathrm{intal}} \circ \mathrm{Sp}_2(\chi)$  for any arithmetic specialization  $\lambda$  of  $h^{\mathrm{ord}}$  with  $\lambda(\mathfrak{a}) = 0$ .*
- (3) *If  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is decomposable, then there exist  $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}$ -valued characters  $\chi_1, \chi_2$  of  $W_{\ell}$  such that  $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $(\chi_1 \oplus \chi_2) \otimes \overline{\mathcal{Q}}(\mathfrak{a})$  and  $\mathrm{WD}(\rho_{f_{\lambda}}|_{W_{\ell}})^{\mathrm{Fr}\text{-ss}}$  is isomorphic to  $\lambda^{\mathrm{intal}} \circ (\chi_1 \oplus \chi_2)$  for any arithmetic specialization  $\lambda$  of  $h^{\mathrm{ord}}$  with  $\lambda(\mathfrak{a}) = 0$ .*

*Consequently, the notion of automorphic types of minimal prime ideals of  $h^{\mathrm{ord}}$  is well-defined.*

*Proof.* Note that  $\mathrm{tr} \rho_{\mathfrak{a}}$  is a pseudorepresentation of  $G_{\mathbb{Q}}$  with values in  $\mathcal{R}(\mathfrak{a})$  and  $\rho_{\mathfrak{a}}$  is irreducible. For any prime  $\mathfrak{p}$  of  $\mathcal{R}(\mathfrak{a})$ , the ring  $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}}$  is Noetherian and hence  $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}}^h$  is Noetherian (see [Sta14, Tag 06LJ]). So  $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}}^h$  is Hausdorff by Krull intersection theorem (see [Mat89, Theorem 8.10]). Note that by Grothendieck's monodromy theorem ([ST68, p. 515–516]),

$\rho_{\mathfrak{a}}|_{G_\ell}$  is monodromic (see the proof of [BC09, Lemma 7.8.14]). For each arithmetic specialization  $\lambda$  of  $h^{\text{ord}}$  with  $\lambda(\mathfrak{a}) = 0$ ,  $\rho_{f_\lambda}$  is an irreducible  $G_F$ -representation (by [Rib77, Theorem 2.3]) over an algebraic closure of the residue field of  $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}_\lambda}$ ,  $\text{tr} \rho_{f_\lambda}$  is equal to  $\lambda \circ \text{tr} \rho_{\mathfrak{a}}$  and  $\rho_{f_\lambda}|_{G_\ell}$  is pure (by [Car86]). So by theorem 5.6, we get part (1), (2), (3). Since local-global compatibility holds for cusp forms (by [Car86]) and each minimal prime ideal of  $h^{\text{ord}}$  is contained in the kernel of some arithmetic specialization of  $h^{\text{ord}}$  (as  $h^{\text{ord}}$  is a finite type  $\mathbb{Z}_p[[X]]$ -module), the final part follows.  $\square$

**7.1.2. Automorphic representations for definite unitary groups.** Let  $F$  be a CM field,  $F^+$  be its maximal totally real subfield. Let  $n \geq 2$  be an integer and assume that if  $n$  is even, then  $n[F^+ : \mathbb{Q}]$  is divisible by 4. Let  $\ell > n$  be a rational prime and assume that every prime of  $F^+$  lying above  $\ell$  splits in  $F$ . Let  $K$  be a finite extension of  $\mathbb{Q}_\ell$  in  $\overline{\mathbb{Q}_\ell}$  which contains the image of every embedding  $F \hookrightarrow \overline{\mathbb{Q}_\ell}$ . Let  $S_\ell$  denote the set of places of  $F^+$  above  $\ell$ . Let  $R$  denote a finite set of finite places of  $F^+$  disjoint from  $S_\ell$  and consisting of places which split in  $F$ . For each place  $v \in S_\ell \cup R$ , choose once and for all a place  $\tilde{v}$  of  $F$  lying above  $v$ . For  $v \in R$ , let  $\text{Iw}(\tilde{v})$  be the compact open subgroup of  $\text{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$  and  $\chi_v$  be the character as in [Ger10, §2.1, 2.2].

Let  $G$  be the reductive algebraic group over  $F^+$  as in [Ger10, §2.1]. For each dominant weight  $\lambda$  (as in [Ger10, Definition 2.2.3]) for  $G$ , the group  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$  acts on the spaces  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$ ,  $S_{\lambda, \{\chi_v\}}^{\text{ord}}(\mathcal{O}_K)$  (as in [Ger10, Definition 2.2.4, 2.4.2]). For an irreducible constituent  $\pi$  of the  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$ , let  $\text{WBC}(\pi)$  denote the weak base change of  $\pi$  to  $\text{GL}_n(\mathbb{A}_F)$  (which exists by [Lab11, Corollaire 5.3]) and let  $r_\pi : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$  (as in [Ger10, Proposition 2.7.2]) denote the unique (up to equivalence) continuous semisimple representation attached to  $\text{WBC}(\pi)$  via [CH13, Theorem 3.2.5].

An irreducible constituent  $\pi$  of the  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$  is said to be an *ordinary automorphic representation for  $G$*  if  $\pi^{U^{(b,c)}} \cap S_{\lambda, \{\chi_v\}}^{\text{ord}}(U^{(b,c)}, \mathcal{O}_K) \neq 0$  for some integers  $0 \leq b \leq c$  (see [Ger10, Definition 2.2.4, §2.3] for details). Let  $U$  be a compact open subgroup of  $G(\mathbb{A}_{F^+}^\infty)$ ,  $T$  be a finite set of finite places of  $F^+$  containing  $R \cup S_\ell$  and such that every place in  $T$  splits in  $F$  (see [Ger10, §2.3]). Let  $\mathbb{T}^{\text{ord}}$  denote the universal ordinary Hecke algebra  $\mathbb{T}_{\{\chi_v\}}^{T, \text{ord}}(U(\mathfrak{l}^\infty), \mathcal{O}_K)$  (as in [Ger10, Definition 2.6.2]). Let  $\Lambda$  be the completed group algebra as in [Ger10, Definition 2.5.1]. By definition of  $\mathbb{T}^{\text{ord}}$ , it is equipped with a  $\Lambda$ -algebra structure and is finite over  $\Lambda$ . An  $\mathcal{O}_K$ -algebra homomorphism  $f : A \rightarrow \overline{\mathbb{Q}_\ell}$  is said to be an *arithmetic specialization* of a finite  $\Lambda$ -algebra  $A$  if  $\ker(f|_\Lambda)$  is equal to the prime ideal  $\wp_{\lambda, \alpha}$  (as in [Ger10, Definition 2.6.3]) of  $\Lambda$  for some dominant weight  $\lambda$  for  $G$  and a finite order character  $\alpha : T_n(\mathfrak{l}) \rightarrow \mathcal{O}_K^\times$ . By [Ger10, Lemma 2.6.4], each arithmetic specialization  $\eta$  of  $\mathbb{T}^{\text{ord}}$  determines an ordinary automorphic representation  $\pi_\eta$  for  $G$ . An arithmetic specialization  $\eta$  of  $\mathbb{T}^{\text{ord}}$  is said to be *stable* if  $\text{WBC}(\pi_\eta)$  is cuspidal.

Let  $\mathfrak{m}$  be a non-Eisenstein maximal ideal of  $\mathbb{T}^{\text{ord}}$  (in the sense of [Ger10, §2.7]). Let  $r_{\mathfrak{m}}$  denote the representation of  $G_{F^+}$  as in [Ger10, Proposition 2.7.4]. Then by restricting it to  $G_F$  and then composing with the projection  $\text{GL}_n(\mathbb{T}_{\mathfrak{m}}^{\text{ord}}) \times \text{GL}_1(\mathbb{T}_{\mathfrak{m}}^{\text{ord}}) \rightarrow \text{GL}_n(\mathbb{T}_{\mathfrak{m}}^{\text{ord}})$ , we get a continuous representation  $G_F \rightarrow \text{GL}_n(\mathbb{T}_{\mathfrak{m}}^{\text{ord}})$  which is denoted by  $r_{\mathfrak{m}}$  by abuse of notation. Since  $\mathfrak{m}$  is non-Eisenstein, the  $G_F$ -representations  $\eta \circ r_{\mathfrak{m}}$  and  $r_{\pi_\eta}$  are isomorphic for any arithmetic specialization  $\eta$  of  $\mathbb{T}_{\mathfrak{m}}^{\text{ord}}$  (by [Ger10, Proposition 2.7.2, 2.7.4]).

**Definition 7.3.** Let  $w$  be a finite place of  $F$  not lying above  $\ell$  and  $\mathfrak{a}$  be a minimal prime of  $\mathbb{T}^{\text{ord}}$ . If the maximal ideal of  $\mathbb{T}^{\text{ord}}$  containing  $\mathfrak{a}$  is non-Eisenstein and some stable arithmetic specialization of  $\mathbb{T}^{\text{ord}}$  vanishes on  $\mathfrak{a}$ , then the automorphic type of  $\mathfrak{a}$  at  $w$  is defined to be the unordered tuple  $\text{AT}_w(\mathfrak{a})$  if the automorphic types of  $\text{WBC}(\pi_\eta)_w$  are equal to  $\text{AT}_w(\mathfrak{a})$  for all stable arithmetic specialization  $\eta$  of  $\mathbb{T}^{\text{ord}}$  with  $\eta(\mathfrak{a}) = 0$ .

**Theorem 7.4.** Let  $w \nmid \ell$  be a finite place of  $F$ ,  $\mathfrak{a}$  be a minimal prime of  $\mathbb{T}_m^{\text{ord}}$  and  $\mathfrak{m}$  be the maximal ideal of  $\mathbb{T}^{\text{ord}}$  containing  $\mathfrak{a}$ . Suppose  $\mathfrak{m}$  is non-Eisenstein. Denote the quotient ring  $\mathbb{T}_m^{\text{ord}}/\mathfrak{a}$  by  $\mathcal{R}(\mathfrak{a})$  and the representation  $r_{\mathfrak{m}} \bmod \mathfrak{a}$  by  $r_{\mathfrak{a}}$ . Then there exist positive integers  $m, t_1 \leq \dots \leq t_m$  and irreducible Frobenius-semisimple representations  $r_1, \dots, r_m$  of  $W_w$  over  $\mathcal{R}(\mathfrak{a})^{\text{intal}}[1/\ell]$  such that  $\text{WD}(r_{\mathfrak{a}}|_{W_w})^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(r_i)$  over  $\overline{\mathcal{Q}}(\mathcal{R}(\mathfrak{a}))$  and  $\text{WD}(r_{\pi_\eta}|_{W_w})^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\eta^{\text{intal}} \circ r_i)$  for any stable arithmetic specialization  $\eta$  of  $\mathcal{R}(\mathfrak{a})$ . Consequently, the notion of local automorphic types of minimal prime ideals of  $\mathbb{T}^{\text{ord}}$  is well-defined. Moreover, two minimal prime ideals of  $\mathbb{T}^{\text{ord}}$  are contained in two non-Eisenstein maximal ideals and are both contained in the kernel of a stable arithmetic specialization of  $\mathbb{T}^{\text{ord}}$  only if their automorphic types at any finite place  $v \nmid \ell$  of  $F$  are the same.

*Proof.* If  $\pi$  is an irreducible constituent of the  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\widetilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}}_\ell)$  such that  $\text{WBC}(\pi)$  is cuspidal, then for any finite place  $w$  of  $F$  not dividing  $\ell$ ,  $r_\pi|_{G_w}$  is pure by [Car12, Theorem 1.1, 1.2] and the proofs of theorem 5.8, corollary 5.9 of *loc. cit.* Note that  $r_{\mathfrak{a}}|_{W_w}$  is monodromic by Grothendieck's monodromy theorem (see [ST68, p. 515–516]). So theorem 4.1 (or theorem 5.6) gives the first part. By [Car12, Theorem 1.1] on local-global compatibility of cuspidal automorphic representations for  $\text{GL}_n$ , the notion of local automorphic types is well-defined. Then the rest follows from the first part.  $\square$

**7.2. Eigenvarieties.** Let  $X$  be a rigid analytic space over a finite extension of  $\mathbb{Q}_p$ . If  $z$  is an element of  $X(\overline{\mathbb{Q}}_p)$ , then the map  $\mathcal{O}(X) \rightarrow \overline{\mathbb{Q}}_p$  is denoted by  $\text{ev}_{zX}$ . The restriction map between the global sections of two admissible open subsets  $U \supset V$  of  $X$  is denoted by  $\text{res}_{UV}$ .

Let  $E/\mathbb{Q}$  be an imaginary quadratic field and  $G$  denote the definite unitary group  $U(m)$  (as in [BC09, §6.2.2]) in  $m \geq 1$  variables. We assume that  $p$  splits in  $E$ . Let  $\mathcal{H}$  denote the Hecke algebra as in [BC09, §7.2.1]. Let  $\mathcal{Z}_0 \subset \text{Hom}_{\text{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^m$  be the set of pairs  $(\psi_{(\pi, \mathcal{R})}, \underline{k})$  associated to the  $p$ -refined automorphic representations  $(\pi, \mathcal{R})$  of any weight  $\underline{k}$  (see [BC09, §7.2.2, 7.2.3]). Let  $e$  be the idempotent as in [BC09, §7.3.1] and let  $\mathcal{Z}_e \subset \mathcal{Z}_0$  denote the subset of  $(\psi_{(\pi, \mathcal{R})}, \underline{k})$  such that  $e(\pi^p) \neq 0$ . We assume that  $\mathcal{Z}_e$  is nonempty. Then by [BC09, §7.3], there exists an eigenvariety for  $\mathcal{Z}_e$ , *i.e.*, there exist a reduced rigid analytic space  $X$  over a finite extension  $L$  of  $\mathbb{Q}_p$ , a ring homomorphism  $\psi : \mathcal{H} \rightarrow \mathcal{O}(X)$ , an analytic map  $\omega : X \rightarrow \text{Hom}((\mathbb{Z}_p^\times)^m, \mathbb{G}_m^{\text{rig}}) \times_{\mathbb{Q}_p} L$  and an accumulation and Zariski-dense subset  $Z$  of  $X(\overline{\mathbb{Q}}_p)$  such that conditions (i), (ii), (iii) of [BC09, Definition 7.2.5] hold. In particular,  $z \mapsto (\text{ev}_{zX} \circ \psi, \omega(z))$  induces a bijection  $Z \xrightarrow{\sim} \mathcal{Z}_e$ . The set  $Z$  is called the set of *arithmetic points* of  $X$ . Let  $Z_{\text{reg}} \subset Z$  be the subset of points parametrizing the  $p$ -refined automorphic representations  $(\pi, \mathcal{R})$  such that  $\pi_\infty$  is regular and the semisimple conjugacy class of  $\pi_p$  has  $m$  distinct eigenvalues (see [BC09, §7.5.1]). By [BC09, Lemma 7.5.3],  $Z_{\text{reg}}$  is a Zariski-dense subset of  $X$ . For each  $z \in Z$ , we fix a  $p$ -refined automorphic representation  $\pi_z$  of  $U(m)$  such that  $z$  corresponds to  $\pi_z$  under the bijection  $Z \xrightarrow{\sim} \mathcal{Z}_e$ . For each  $z \in Z_{\text{reg}}$ , let  $\rho_{z,p} : G_E \rightarrow \text{GL}_m(\overline{\mathbb{Q}}_p)$  denote the unique (up to equivalence) continuous semisimple representation attached to  $\text{WBC}(\pi_z)$  via [CH13, Theorem 3.2.5]. By [BC09, Proposition

7.5.4], there exists a pseudorepresentation  $T : G_E \rightarrow \mathcal{O}(X)$  such that  $\text{ev}_{zX} \circ T = \text{tr} \rho_{z,p}$  for all  $z \in Z_{\text{reg}}^{\text{st}}$ . Let  $Z_{\text{reg}}^{\text{st}}$  denote the set of points  $z \in Z_{\text{reg}}$  such that  $\text{WBC}(\pi_z)$  is cuspidal. For  $z \in Z_{\text{reg}}^{\text{st}}$ , the Galois representation  $\rho_{z,p}$  is expected to be irreducible. It is known when  $m \leq 3$  by [BR92] and in many cases when  $m = 4$  by an unpublished work of Ramakrishnan. By [PT13, Theorem D], it is known for infinitely many primes  $p$ .

**Definition 7.5.** *Let  $w$  be a finite place of  $E$  not lying above  $p$  and  $Y_0$  be an irreducible component of  $X$  such that  $Z_{\text{reg}}^{\text{st}} \cap Y_0$  is nonempty. The automorphic type of  $Y_0$  at  $w$  is defined to be the unordered tuple  $\text{AT}_w(Y_0)$  if the automorphic types of  $\text{WBC}(\pi_z)_w$  are equal to  $\text{AT}_w(Y_0)$  for all  $z \in Z_{\text{reg}}^{\text{st}} \cap Y_0$ .*

Let  $\xi : \tilde{X} \rightarrow X$  be a normalization of  $X$ . Let  $C$  be a connected component of  $\tilde{X}$  and  $Y$  be the irreducible component  $\xi(C)$  (together with its canonical structure of reduced rigid space) of  $X$ . By [Con99, Lemma 2.2.1 (2)], the map  $\xi|_C : C \rightarrow Y$  is a normalization. For each  $x \in X(\overline{\mathbb{Q}}_p) \cap Y$ , we fix a point  $\tilde{x}$  in  $C(\overline{\mathbb{Q}}_p)$  which goes to  $x$  under the map  $C(\overline{\mathbb{Q}}_p) \rightarrow Y(\overline{\mathbb{Q}}_p)$ .

**Theorem 7.6.** *Let  $w \nmid p$  be a finite place of  $E$ . Suppose that the intersection of  $Z_{\text{reg}}^{\text{st}}$  with any irreducible component of  $X$  is nonempty and for any  $z \in Z_{\text{reg}}^{\text{st}}$ , the Galois representation  $\rho_{z,p}$  is irreducible. Then there exist positive integers  $n, t_1, \dots, t_n$ , irreducible Frobenius-semisimple representations  $r_1, \dots, r_n$  of  $W_w$  over  $\mathcal{O}(C)^{\text{intal}}$  such that the following hold.*

- (1) *The polynomial  $(\text{Eul}(\rho_C, N_C))^{-1}$  has coefficients in  $\mathcal{O}(C)^{\text{intal}}$  and  $\text{res}_{\tilde{X}C} \circ \xi \circ T|_{W_w}$  is equal to the trace of  $(\rho_C, N_C)$  where*

$$(\rho_C, N_C) := \bigoplus_{i=1}^n \text{Sp}_{t_i}(r_i)_{/\mathcal{O}(C)^{\text{intal}}}.$$

- (2) *If  $z \in Z_{\text{reg}}^{\text{st}} \cap Y$ , or more generally if  $z \in Z_{\text{reg}} \cap Y$  such that  $\rho_{z,p}$  is irreducible and  $\rho_{z,p}|_{W_w}$  is pure, then for any arbitrary lift  $\text{ev}_{\tilde{z}C}^{\text{intal}}$  of  $\text{ev}_{\tilde{z}C}$ , there is an isomorphism*

$$\text{WD}(\rho_{z,p}|_{W_w})^{\text{Fr-ss}} \simeq \text{ev}_{\tilde{z}C}^{\text{intal}} \circ (\rho_C, N_C)$$

and

$$\text{ev}_{\tilde{z}C}^{\text{intal}}(\text{Eul}(\rho_C, N_C)) = \text{Eul}_w(\rho_{\pi_{z,p}}).$$

- (3) *Let  $V$  be a nonempty connected admissible open subset of  $C$  and  $(\rho_V, N_V) : W_w \rightarrow \text{GL}_m(\mathcal{O}(V)^{\text{intal}})$  be a Weil-Deligne representation such that  $\text{res}_{CV} \circ \text{res}_{\tilde{X}C} \circ \xi \circ T = \text{tr} \rho_V$  and  $f_V \circ (\rho_V, N_V)$  is pure for some  $\mathbb{Z}_p$ -algebra homomorphism  $f_V : \mathcal{O}(V)^{\text{intal}} \rightarrow \overline{\mathbb{Q}}_p$ . Then for any arbitrary lift  $\text{res}_{CV}^{\text{intal}}$  of  $\text{res}_{CV}$ , there is an isomorphism*

$$((\rho_V, N_V) \otimes_{\mathcal{O}(V)} \overline{\mathbb{Q}}(\mathcal{O}(V)))^{\text{Fr-ss}} \simeq (\text{res}_{CV}^{\text{intal}} \circ (\rho_C, N_C)) \otimes_{\mathcal{O}(V)^{\text{intal}}} \overline{\mathbb{Q}}(\mathcal{O}(V)).$$

Consequently, the notion of local automorphic types of irreducible components of  $X$  is well-defined. Moreover, two irreducible components of  $X$  intersect at a point of  $Z_{\text{reg}}^{\text{st}}$  only if their local automorphic types at any finite place of  $E$  outside  $p$  are the same.

*Proof.* By [Con99, Lemma 2.1.4],  $\mathcal{O}(C)$  is an integral domain over  $\mathbb{Z}_p$ . By [Tay91, Theorem 1],  $\text{res}_{\tilde{X}C} \circ \xi \circ T$  is equal to the trace of a representation  $\sigma$  of  $G_E$  over  $\overline{\mathbb{Q}}(\mathcal{O}(C))$ . Since  $Z_{\text{reg}}^{\text{st}} \cap Y$  is nonempty and the Galois representations attached to its points are irreducible, the representation  $\sigma$  is irreducible. Let  $z$  be a point in  $Z_{\text{reg}}^{\text{st}} \cap Y$ . We choose a connected affinoid neighbourhood  $U_{\tilde{z}}$  of  $\tilde{z}$ . Note that  $U_{\tilde{z}}$  is contained in  $C$  and the map  $\text{res}_{CU_{\tilde{z}}}$  is injective by [Con99, Lemma 2.1.4]. The point  $\tilde{z}$  defines a maximal ideal  $\mathfrak{m}_{\tilde{z}}$  of  $\mathcal{O}(U_{\tilde{z}})$ . The

localization of  $\mathcal{O}(U_{\bar{z}})$  at  $\mathfrak{m}_{\bar{z}}$  is Noetherian and hence the Henselization of this localization is Hausdorff by Krull intersection theorem (see [Mat89, Theorem 8.10]). Note that  $\rho_{z,p}$  is an irreducible representation of  $G_E$  over an algebraic closure of the residue field of  $\mathcal{O}(U_{\bar{z}})_{\mathfrak{m}_{\bar{z}}}$  and has trace equal to  $\pi_{\mathfrak{m}_{\bar{z}}} \circ \text{res}_{CU_{\bar{z}}} \circ (\text{res}_{\tilde{X}_C} \circ \xi \circ T)$ . By [Tay91, Theorem 1], there exists a semisimple  $G_E$ -representation  $\tilde{\rho}_{U_{\bar{z}}}$  over  $\bar{Q}(\mathcal{O}(U_{\bar{z}}))$  such that  $\text{tr} \tilde{\rho}_{U_{\bar{z}}} = \text{res}_{CU_{\bar{z}}} \circ \text{res}_{\tilde{X}_C} \circ \xi \circ T$  and the restriction of  $\tilde{\rho}_{U_{\bar{z}}}$  to  $W_w$  is monodromic by [BC09, Lemma 7.8.11, 7.8.14]. So  $\sigma|_{W_w}$  is monodromic. Since  $\rho_{z,p}$  is irreducible,  $\tilde{\rho}_{U_{\bar{z}}}$  is also irreducible. By Zorn's lemma, we have  $\text{ev}_{\tilde{Z}C}^{\text{intal}} = \text{ev}_{\tilde{Z}U_{\bar{z}}}^{\dagger} \circ \text{res}_{CU_{\bar{z}}}^{\dagger}$  for some lifts  $\text{ev}_{\tilde{Z}U_{\bar{z}}}^{\dagger}, \text{res}_{CU_{\bar{z}}}^{\dagger}$  of  $\text{ev}_{\tilde{Z}U_{\bar{z}}}, \text{res}_{CU_{\bar{z}}}$  respectively. So by theorem 5.6, we get part (1) and (2). Using theorem 5.4, we get part (3). By [Car12, Theorem 1.1] on local-global compatibility of cuspidal automorphic representations for  $\text{GL}_n$ , the notion of local automorphic types is well-defined. Then the rest follows.  $\square$

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